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The Dirichlet Problem for the Kohn Laplacian on the Heisenberg Group, II

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Let $L = \sum_{j=1}^{m} X_j^2$ be sum of squares of vector fields in $\mathbb{R}^n$ satisfying a Hörmander condition of order 2: span$\{X_j, [X_i, X_j]\}$ is the full tangent space at each point. A point $x \in \partial D$ of a smooth domain D is *characteristic* if $X_1,..., X_m$ are all tangent to ∂D at x. We prove sharp estimates in non-isotropic Lipschitz classes for the Dirichlet problem near (generic) isolated characteristic points in two special cases: (a) The Grushin operator $\partial^2/\partial x^2 + x^2\, \partial^2/\partial t^2$ in $\mathbb{R}^2$. (b) The real part of the Kohn Laplacian on the Heisenberg group $\sum_{j=1}^{n} (\partial/\partial x_j + 2y_j\, \partial/\partial t)^2 + (\partial/\partial y_j - 2x_j\, \partial/\partial t)^2$ in $\mathbb{R}^{2n+1}$. In contrast to non-characteristic points, C^∞ regularity may fail at a characteristic point. The precise order of regularity depends on the shape of ∂D at x.

Contents.

1. INTRODUCTION[1]

Let $X_1,..., X_m$ be smooth real vector fields on $\mathbb{R}^n$ satisfying a Hörmander-type condition of step 2, namely, $X_1,..., X_m$, $[X_i, X_j]$, $i, j = 1,..., m$, span the tangent space at each point. Let D be a smooth bounded domain in $\mathbb{R}^n$. Denote $\mathscr{L} = -\sum_{j=1}^{m} X_j^2$. Kohn [11] and Hörmander [8] have proved that for $\phi \in C_0^\infty(D)$,

$$\operatorname{Re} \int (\mathscr{L}\phi)\, \bar{\phi} \geqslant C_1 \|\phi\|_{H_{1/2}}^2 - C_2 \|\phi\|_{L^2(D)}^2 \tag{1.1}$$

(H_s is the Sobolev space of order s.) It follows that the Dirichlet problem (see [12])

$$\mathscr{L}u = f \text{ in } D; \qquad u\,|_{\partial D} = g \tag{1.2}$$

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[1] There is an error in 3.3 that affects Sections 3 and 5. A correction can be found at the end of the paper.

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has a generalized solution for $f \in C^\infty(\bar{D})$, $g \in C^\infty(\partial D)$ (modulo a finite number of compatibility conditions).

A point x of ∂D is called *characteristic* if the principal symbol of $\mathscr{L}$ annihilates the normal to ∂D at x. Kohn and Nirenberg [12] showed that under very general circumstances if an operator satisfies an estimate like (1.1), then the solution u to the Dirichlet problem is smooth up to the boundary at all *non*-characteristic points. The study of characteristic points for second-order operators with semidefinite principal symbol was initiated by Fichera [3], and extended Oleinik [17] and Kohn and Nirenberg [13]. (See [13] for further references.) However, their results concerning boundary regularity do not apply to the sort of characteristic point that arises for an operator $\mathscr{L}$ of Hörmander type.

In Part I [9] we examined non-characteristic boundary points. In Part II we shall be concerned with estimates for u near characteristic points in two special cases:

A. The Grushin operator $L = -\left(\dfrac{\partial^2}{\partial x^2} + \left(x\dfrac{\partial}{\partial t}\right)^2\right)$ on $\mathbb{R}^2$.

B. $\mathscr{L}_0 = -\dfrac{1}{4}\sum_{j=1}^{2n} X_j^2$ on $\mathbb{R}^{2n+1}$, where $X_j = \dfrac{\partial}{\partial x_j} + 2y_j\dfrac{\partial}{\partial t}$,

$$X_{j+n} = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\ j = 1,\dots, n,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

The operator $\mathscr{L}_0$ represents the real part of the Kohn Laplacian on the *Heisenberg group* (see [11, 14]).

Let us give a rough statement of our main result. We will measure smoothness in "non-isotropic" Lipschitz (or Hölder) classes Γ_β, $0 < \beta < \infty$ (See Section 4, [4; 9; 16].) These classes are suited to $\mathscr{L}$ in the same way that the usual scale of Lipschitz classes Λ_β are suited to elliptic operators. Suppose that in the Dirichlet problem (1.2), f is of class $\Gamma_{\beta-2}$ on ∂D and g is of class Γ_β. Suppose also that the characteristic point x_0 has a certain generic type (called strongly isolated; see Section 4). Then u is of class Γ_β only if $\beta < \beta_0$, where β_0 is a critical index depending only of the Hessian, $X_iX_jr(x_0)$, $i, j = 1,\dots, m$, at x_0 of a defining function r for D. (The solution u is constructed by the Perron method, not by L^2 methods.) The main idea of Part II is to introduce appropriate polar coordinates. This technique was first exploited by Kondratiev [14] in the classical Dirichlet problem.

The plan of the paper is as follows. In the second section we treat the Grushin operator. The point is to illustrate the idea of non-isotropic dilation and the close analogy between a characteristic point and the vertex in the classical Dirichlet problem for a cone (see [7, 14, 15]). The rest of the paper is devoted to extending these ideas from the Grushin operator to $\mathscr{L}_0$. This extension is far from complete. In particular, it would be useful to obtain similar estimates for the full Kohn Laplacian, $\mathscr{L}_\alpha = \mathscr{L}_0 + i\alpha(\partial/\partial t)$ for $\alpha = n - 2q$, $q = 1, 2, \ldots, n-1$.

In the third section we prove a small amount of Hölder continuity for solutions to the homogeneous ($f = 0$) Dirichlet problem for $\mathscr{L}_0$ and exhibit explicit examples of solutions with singularities. In the fourth section we recall the definitions of Lipschitz classes Γ_β and describe their restriction to ∂D near (strongly isolated) characteristic points. In the final section we use dilation invariance, the results of Part I [9], the weak maximum principle, and a strict maximum principle due to Bony [1] to deduce regularity a characteristic points for $\mathscr{L}_0$. The use of the (weak) maximum principle is what limits our regularity theorem to $\mathscr{L}_\alpha$ for $\alpha = 0$ only. More generally, one might hope to prove regularity in the Dirichlet problem for the Kohn Laplacian, $\Box_b$, on CR manifolds and to attack other boundary value problems for $\Box_b$ such as the $\bar{\partial}_b$-Neumann problem.

This article is the sequel to [9], which will be referred to throughout as Part I. The notations I.3.9 and Theorem I.3.9, for example, refer to Theorem 3.9 of Part I. I would like to reiterate my thanks to my adviser, E. M. Stein, for his valuable suggestions.

2. The Grushin Operator

Associated to the Grushin operator $L = -\partial^2/\partial x^2 - x^2\partial^2/\partial t^2$ is a natural dilation $\delta(x, t) = (\delta x, \delta^2 t)$. Denote $f_\delta(x, t) = f(\delta x, \delta^2 t)$, then $L(f_\delta) = \delta^2 (Lf)_\delta$. Define a distance $d((x, t), (x', t')) \equiv |x - x'| + \min(|t - t'|/|x|, |t - t'|^{1/2})$. Then

$$d((x, t), (x', t')) \leqslant C d((x', t'), (x, t)),$$
$$d((x_1, t_1), (x_3, t_3)) \leqslant C(d(x_1, t_1), (x_2, t_2)) + d((x_2, t_2), (x_3, t_3))$$

(symmetry and triangle inequality). Denote $\Gamma_\beta = \{f \in C(\mathbb{R}^2) \mid$ for all $\delta > 0$, $a \in \mathbb{R}^2$, there exists $P(b)$ a polynomial such that $|f(b) - P(b)| < A\delta^\beta$ whenever $d(a, b) < \delta\}$. The Γ_β norm of f is $\sup |f| +$ the smallest A above. For information about Γ_β see [16]. In particular we note that $\|f\|_{\Gamma_{\beta+1}}$ is equivalent to $\|(\partial/\partial x) f\|_{\Gamma_\beta} + \|x(\partial/\partial t) f\|_{\Gamma_\beta} + \|f\|_{L^\infty}$. The spaces Γ_β, $0 < \beta < \infty$, form a scale under real interpolation.

Let D be a smooth domain in the (x, t) plane. A characteristic point of ∂D is a point for which $\partial/\partial x$ and $x(\partial/\partial t)$ are tangent to ∂D. By a translation in t we may as well assume that the characteristic point is $(0, 0)$. Then ∂D is horizontal at the origin and has the equation

$$t = Mx^2 + g(x), \qquad \text{where } g = O(x^3).$$

The region $D_M = \{(x, t) \mid t > Mx^2\}$ is tangent to D at $(0, 0)$ to third order and D_M is invariant under dilation. We can therefore solve the Dirichlet problem for L in D_M by separation of variables, using polar coordinates. The hope is that the behavior of solutions to the Dirichlet problem in D near $(0, 0)$ is similar to the behavior in D_M. This procedure has been carried out for the usual Laplacian in dilation invariant regions relative to usual dilations, in other words, cones (see [7, 14, 15]).

Define polar coordinates by

$$\rho = (x^4 + 4t^2)^{1/4}, \qquad \tau = \cos\theta = 2t/\rho^2.$$

$$L = 4(1-\tau^2)^{1/2}\left\{(1-\tau^2)\rho^{-2}\frac{\partial^2}{\partial\tau^2} - \frac{3}{2}\tau\rho^{-2}\frac{\partial}{\partial\tau} + \frac{1}{4}\frac{\partial^2}{\partial\rho^2} + \frac{1}{2}\rho^{-1}\frac{\partial}{\partial\rho}\right\}. \tag{2.1}$$

Hence, $L(g_\nu(\tau)\rho^\nu) = 0$ if and only if

$$\left[(1-\tau^2)\frac{d^2}{d\tau^2} - \frac{3}{2}\tau\frac{d}{d\tau} + (\nu/2)(\nu/2 + 1/2)\right] g_\nu(\tau) = 0. \tag{2.2}$$

This is a Jacobi equation [18, p. 60]. One solution is usually denoted $P_{\nu/2}^{(1/4)}(\tau)$. It is an ultraspherical polynomial when ν is an even integer. Two independent solutions are given by

$$g_0(x, t; \nu) = F_0(\tfrac{1}{2}(1-\tau), \nu)$$
$$g_1(x, t; \nu) = (\operatorname{sgn} x)\,|\tfrac{1}{2}(1-\tau)|^{1/4}\,F_1(\tfrac{1}{2}(1-\tau), \nu),$$

where F_0 and F_1 are hypergeometric series:

$$F_0(z, \nu) = F((\nu+1)/2, \nu/2; 3/4; z),$$
$$F_1(z, \nu) = F(\nu/2 + 3/4, -\nu/2 + 1/4; 5/4; z).$$

Note that $(\operatorname{sgn} x)\,|\tfrac{1}{2}(1-\tau)|^{1/4} = x(1 + 2t\rho^{-2})^{-1/4}$ is smooth away from $(0, 0)$. Denote the negative t axis by $L_1 = \{(0, t) \mid t \leqslant 0\}$. For $(x, t) \in \mathbb{R}^2 \backslash L_1$, $z = \frac{1}{2}(1+\tau) < 1$, so the hypergeometric series are convergent. Thus g_0 and g_1 are smooth in $\mathbb{R}^2 \backslash L_1$. They satisfy $L(g_0\rho^\nu) = L(g_1\rho^\nu) = 0$ and are the unique even and odd functions of x (up to multiplies) satisfying this property.

Define coordinates $\phi : \mathbb{R} \to \partial D_M$ by $\phi(y) = (1 + 4M^2)^{-1/4}(y, My^2)$. (Note that $\rho(\phi(y)) = |y|$.) Suppose that the Poisson kernel for D_M exists. More precisely, suppose there is $P(x, t; y)$ such that if $f \in C_0^\infty(\mathbb{R})$ and $P(f) = u(x, t) = \int_{-\infty}^{\infty} P(x, t; y)\, f(y)\, dy$, then $Lu = 0$ in D_M, $\lim_{t\to\infty} u(x, t) = 0$, u is continuous in $\bar{D}_M$ and $u(\phi(y))=f(y)$. We will give a formal argument to show that Mellin transform of P is given in terms of g_0 and g_1. We will then *define* P as the inverse Mellin transform of the correct symbol and prove in this way existence and estimates for the Poisson kernel.

Formally, let

$$S_0(x, t; \nu) = \int_{-\infty}^{\infty} P(x, t;\, y)\,|y|^{\nu}\, dy,$$

$$S_1(x, t; \nu) = \int_{-\infty}^{\infty} P(x, t;\, y)\,|y|^{\nu}(\operatorname{sgn} y)\, dy.$$

We expect dilation invariance for P:

$$P(\delta x, \delta^2 t; \delta y)\, d(\delta y) = P(x, t;\, y)\, dy.$$

Hence, $S_0(\delta x, \delta^2 t; \nu) = \delta^{\nu} S_0(x, t; \nu)$ and similarly for S_1. Therefore, we expect

$$S_j(x, t; \nu) = h_j(x, t; \nu)\,\rho^{\nu}. \qquad j = 1, 2,$$

where h_j is some multiple of g_j. Since $P(\rho^{\nu})$ has boundary values ρ^{ν}, we expect that $S_0(x, t; \nu) = \rho^{\nu}$, $S_1(x, t; \nu) = (\operatorname{sgn} x)\,\rho^{\nu}$ for $(x, t) \in \partial D_M$. This uniquely specifies S_0 and S_1. Denote

$$c_j(M, \nu) = g_j(x, Mx^2; \nu). \qquad \text{(This is independent of } x.)$$

$$S_j(x, t; \nu) = \sigma_j(\theta, \nu)\,\rho^{\nu}. \qquad (\cos\theta = \tau)$$

$$\sigma_j(\theta, \nu) = \frac{g_j(x, t; \nu)}{c_j(M, \nu)}.$$

Let θ_0 be defined by $0 < \theta_0 < \pi$ and $\cos\theta_0 = 2M/(1 + 4M^2)^{1/2}$. Then

$$D_M = \{(\rho, \theta) \mid |\theta| < \theta_0\}.$$

Several candidates for the Poisson kernel are given by the integral formula

$$P(x, t;\, y) = \frac{1}{2\pi}\, y^{-1} \int_{-\infty}^{\infty} \tfrac{1}{2}(\sigma_0(\theta, \beta + i\eta) + \sigma_1(\theta, \beta + i\eta))(\rho/y)^{\beta + i\eta}\, d\eta. \tag{2.3}$$

In Appendix I we prove several facts about σ_0 and σ_1 that will tell us when (2.3) is convergent.

(a) The zeros of $c_j(M, \nu)$ are real and simple. Thus σ_0 and σ_1 have simple real poles.

(b) If $0 < \nu_0 < \nu_2 < \nu_4 < \cdots$ are the positive zeros of $c_0(M, \nu)$ and $0 < \nu_1 < \nu_3 < \cdots$ are the positive zeros of $c_1(M, \nu)$, then they are interlaced, $0 < \nu_0 < \nu_1 < \nu_2 < \nu_3 < \cdots$. Moreover, the nonpositive zeros of $c_0(M, \nu)$ and $c_1(M, \nu)$ are images of the positive zeros under the mapping $\nu \to -\nu - 1$. In particular, there are no zeros in the interval $-1 \leqslant \nu \leqslant 0$.

(c) $|\sigma_j(\theta, \beta + i\eta)| \leqslant C_\beta e^{-|\theta - \theta_0| |\eta|}$, provided $\beta + i\eta$ is a positive distance from a pole of σ_j.

We will also need a differential version of (c):

(d) $$|\theta - \theta_0|^\delta \left| \frac{\partial^j}{\partial \nu^j} \frac{\partial^k}{\partial \theta^k} \sigma_0(\theta, \nu) \right| \leqslant C(1 + |\eta|)^{-\delta - j + k};$$

where $\nu = \beta + i\eta$, C depends on β, j, k, δ, and $\delta \geqslant 0$.

A similar estimate holds for $\sigma_1(\theta, \nu)$ with a suitable modification to take into account the fact that σ_1 is not smooth in θ as $\theta \to 0$. (σ_1 is smooth near $\theta = 0$ as a function of x and t.)

Estimate (d) expresses the fact that σ_0 and σ_1 are symbols of Poisson type (see [6, p. 88], and Part I, Section 4.)

Denote $I_0 = (-\nu_0 - 1, \nu_0)$; $I_j = (\nu_{j-1}, \nu_j)$, $I_{-j} = (-\nu_j - 1, \ -\nu_{j-1} - 1) =$ 1, 2, 3,... . Properties (a), (b), and (c) show that P_j is well defined:

$$P_j(x, t; y) = \frac{1}{2\pi} y^{-1} \int_{-\infty}^{\infty} \tfrac{1}{2}(\sigma_0(\theta, \beta + i\eta) + \sigma_1(\theta, \beta + i\eta))(\rho/y)^{\beta + i\eta} \, d\eta, \tag{2.3'}$$

with $\beta \in I_j$. By contour integration, the integral is independent of the value of β in I_j.

By estimate (d) we can differentiate under the integral sign for $(x, t) \in D_M$ (i.e., $|\theta| < \theta_0$). The integrand in (2.3′) is annihilated by L because of (2) and the definition of σ_0 and σ_1. Hence, $P_j(x, t; y)$ is annihilated by L for all y.

In order to prove that P_j is a Poisson kernel, it remains to show that $P_j(f)$ tends to f on ∂D_M. (This will only be true for certain f depending on P_j.)

We will deduce from (d) the estimates

$$|P_j| \leqslant C\varepsilon(d((x, t), \phi(y))^{-2} + \rho^{-2})/m_j(\rho/y), \tag{2.4}$$

where $\varepsilon = d((x, t), \partial D_M)$ and

$$m_0(s) = s^{\nu_0} + s^{-\nu_0 - 1},$$
$$m_j(s) = s^{-\nu_j - 1} + s^{-\nu_{j-1} - 1},$$
$$m_{-j}(s) = s^{\nu_j} + s^{\nu_{j-1}}, \qquad j = 1, 2, 3, \ldots .$$

From (2.4) it is easy to see that the integral $P_j(f) = \int_{-\infty}^{\infty} P_j(x, t; y) f(y)\, dy$ converges whenever f satisfies

$$\int_{-\infty}^{\infty} \frac{|f(y)|}{m_j(|y|^{-1})}\, dy < \infty. \tag{2.5}$$

Suppose that $\beta \in I_j$, then $|y|^\beta$ satisfies (2.5) and by Fourier inversion,

$$\int_{-\infty}^{\infty} P_j(x, t; y)\, |y|^\beta\, dy = \sigma_0(\theta, \beta)\, \rho^\beta;$$

$$\int_{-\infty}^{\infty} P_j(x, t; y)\, |y|^\beta\, (\operatorname{sgn} y)\, dy = \sigma_1(\theta, \beta)\, \rho^\beta.$$

Recall that $\sigma_0(\theta_0, \beta) \equiv \sigma_1(\theta_0, \beta) \equiv 1$ and $\rho(\phi(y)) = |y|$. Let f be continuous and satisfy (2.5). Write $f = f_0 + f_1$, where f_0 is even and f_1 is odd. For $\beta \in I_j$, $(x, t) \in D_M$ we can take the limit as $(x, t) \to \phi(y_0)$:

$$\begin{aligned}
\lim P_j f(x, t) &= \lim \int_{-\infty}^{\infty} P_j(x, t; y) f(y)\, dy \\
&= \lim \int_{-\infty}^{\infty} P_j(x, t; y)(f_0(y) - f_0(y_0)\, |y/y_0|^\beta)\, dy \\
&\quad + \lim \int_{-\infty}^{\infty} P_j(x, t; y)(f_1(y) - f_1(y_0)\, |y/y_0|^\beta\, (\operatorname{sgn} y))\, dy \\
&\quad + \lim \int_{-\infty}^{\infty} P_j(x, t; y)\, |y/y_0|^\beta\, dy\, f_0(y_0) \\
&\quad + \lim \int_{-\infty}^{\infty} P_j(x, t; y)\, |y/y_0|^\beta\, (\operatorname{sgn} y)\, dy\, f_1(y_0) \\
&= 0 + 0 + \sigma_0(\theta_0, \beta) f_0(y_0) + \sigma_1(\theta_0, \beta) f_1(y_0) = f(y_0).
\end{aligned}$$

The limits are evaluated using (2.4), (2.5), and the fact that f is continuous.

Note that when $f \in C_0(\mathbb{R})$, f satisfies (2.5) for $j = 0$. Hence, $P_0(f)$ is well-defined. Moreover, by (2.4), $P_0(f)$ tends to zero as $t \to \infty$. Thus by the weak maximum principle $P_0(f)$ is the unique Poisson kernel with the properties given earlier in this section.

Let us now sketch how to deduce estimates on P_j from (d). At the same time we will estimate $X^\gamma P_j(x, t; y)$, where X^γ is a monomial of length $|\gamma|$ in $\partial/\partial x$ and $x(\partial/\partial t)$.

First of all, in the range $y/2 < \rho < 2y$, the estimates are

$$|P_j| \leqslant C\varepsilon d((x,t), \phi(y))^{-2}, \tag{2.6}$$

$$|X^\gamma P_j| \leqslant Cd((x,t), \phi(y))^{-1-|\gamma|}. \tag{2.7}$$

These estimates are like those on the ordinary Poisson kernel in the upper half plane. Observe that $\varepsilon \simeq (\theta-\theta_0)\rho$. From (d) and (2.3′)

$$\begin{aligned} |P_j| &\leqslant Cy^{-1}\int_{|\eta|<|\theta-\theta_0|^{-1}} d\eta + Cy^{-1}\int_{|\eta|>|\theta-\theta_0|^{-1}} |\theta-\theta_0|^{-1}(1+|\eta|)^{-1}\,d\eta \\ &\leqslant Cy^{-1}|\theta-\theta_0|^{-1} \leqslant C\varepsilon^{-1}. \end{aligned}$$

Also, $d((x,t), \phi(y)) \simeq \rho(|\theta-\theta_0| + |\log|\rho/y||)$. A similar splitting $|\eta| > |\log|\rho/y||^{-1}$ and an integration by parts in η gives

$$|P_j| \leqslant Cy^{-1}|\log|\rho/y||^{-1}.$$

These two estimates combine to give the first estimate in (2.7): $|P_j| \leqslant Cd((x,t), \phi(y))^{-1}$. The others are similar.

To obtain (2.6), note that because $\sigma_0(\theta, v) \equiv 1$,

$$\frac{\partial}{\partial v}\sigma_0(\theta, v) = \int_{\theta_0}^{\theta} \frac{\partial}{\partial v}\frac{\partial}{\partial s}\sigma_0(s, v)\,ds.$$

The integrand

$$\frac{\partial}{\partial v}\frac{\partial}{\partial s}\sigma_0(s, v)$$

has the same estimates as σ_0. The same holds for σ_1, hence an integration by parts in η gives $|P_j| \leqslant Cy^{-1}|\theta-\theta_0|(\log|\rho/y|)^{-2}$. This and the bound $C\varepsilon^{-1}$ above give (2.6).

In the range of $\rho < y/2$, shift the contour past the pole at the upper limit of the interval I_j. For I_0, for example, for $v_1 > \beta > v_0$,

$$\begin{aligned} P_0(x, t; y) &= \frac{1}{2\pi}y^{-1}\int_{-\infty}^{\infty} \tfrac{1}{2}(\sigma_0(\theta, \beta+i\eta) + \sigma_0(\theta, \beta+i\eta))(\rho/y)^{\beta+i\eta}\,d\eta \\ &\quad + y^{-1}\frac{g_0(x,t;v_0)}{c_0'(M, v_0)}(\rho/y)^{v_0}. \end{aligned}$$

The second term is the residue at the pole at v_0. $c_0'(M, v)$ denotes

$(d/dv)\, c_0(M, v)$. Since $g_0(x, t; v_0)$ vanishes for $(x, t) \in \partial D_M$, $|g_0(x, t; v_0)| \leqslant C|\theta - \theta_0|$. An integration by parts as in the proof of (2.6) shows likewise,

$$\left| \int_{-\infty}^{\infty} \tfrac{1}{2}(\sigma_0(\theta, \beta + i\eta) + \sigma_1(\theta, \beta + i\eta))(\rho/y)^{i\eta}\, dy \right| \leqslant C|\theta - \theta_0|,$$

for $\rho < y/2$. Hence, since $|(\theta - \theta_0| \simeq \varepsilon\rho^{-1}$,

$$\begin{aligned} |P_0(x, t;\, y)| &\leqq C\varepsilon\rho^{-1}[(\rho/y)^{v_0}y^{-1} + (\rho/y)^{\beta}y^{-1}] \\ &\leqq C\varepsilon\rho^{-2}(\rho/y)^{v_0+1}. \end{aligned}$$

A similar argument using a contour with $\beta < -v_0 - 1$ for $\rho > 2y$ gives $|P_0(x, t;\, y)| \leqslant C\varepsilon\rho^{-2}(\rho/y)^{-v_0}$. Thus

$$|P_0| \leqq C\varepsilon\rho^{-2}/m_0(\rho/y) \qquad \text{for } \rho < \tfrac{1}{2}y \text{ or } \rho > 2y.$$

Differentiation loses a power of ρ each time:

$$|X^\gamma P_0| \leqq C\rho^{-1-|\gamma|}/m_0(\rho/y); \qquad \rho < \tfrac{1}{2}y \text{ or } \rho > 2y.$$

In general we have

$$\begin{aligned} |P_j| &\leqq C\varepsilon(d((x, t), \phi(y))^{-2} + \rho^2)/m_j(\rho/y) \\ |X^\gamma P_j| &\leqq C(d((x, t), \phi(y))^{-1-|\gamma|} + \rho^{-1-|\gamma|})/m_j(\rho/y). \end{aligned} \tag{2.8}$$

It is easy to check that the restriction of Γ_β to ∂D_M is the same as $\Lambda_\beta(\mathbb{R})$, where ∂D_M is identified with $\mathbb{R}$ by ϕ. Indeed, the extension of a function in $\Lambda_\beta(\partial D_M)$ that is constant in the t variable belongs to Γ_β. The restriction of Γ_β to ∂D_M is contained in $\Lambda_\beta(\partial D_M)$ because the tangent to ∂D_M, $\partial/\partial x + 2Mx\, \partial/\partial t$, is a linear combination of the basic vector fields $\partial/\partial x$ and $x\, \partial/\partial t$ and because the restriction of the distance function d to ∂D_m is comparable to ordinary distance on ∂D_M. The restriction of Γ_β to $\bar{D}_M$ will be denoted $\Gamma_\beta(\bar{D}_M)$.

(2.9) THEOREM. *If $f \in \Lambda_\alpha(R)$ has compact support and $v_{j-1} < \alpha < v_j$, then $P(f)$ belongs to $\Gamma_\alpha(\bar{D}_M)$ if and only if f satisfies j compatibility conditions described below. In particular, if $0 < \alpha < v_0$, then $P(f)$ belongs to $\Gamma_\alpha(\bar{D}_M)$ with no compatibility.*

Proof. Let $f = f_0 + f_1$, where f_0 is even and f_1 is odd. Denote the Mellin transform by

$$\tilde{f}_j(v) = \int_0^\infty f_j(y)\, y^{-v} \frac{dy}{y}, \qquad j = 0, 1.$$

$\tilde{f}_j(\nu)$ is meromorphic in $\operatorname{Re}\nu < \alpha$ with poles (possibly) at nonnegative integers. For $-\nu_0 - 1 < \beta < 0$ we can write (by Plancherel's theorem)

$$(Pf)(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\sigma_0(\theta, \beta + i\eta)\, \tilde{f}_0(\beta + i\eta)$$
$$+ \sigma_1(\theta, \beta + i\eta)\tilde{f}_1(\beta + i\eta)]\, \rho^{\beta + i\eta}\, d\eta.$$

Suppose that α is not an integer $k < \alpha < k + 1$. We can extend our estimates to the integer case by real interpolation. Choose β so that $\max(\nu_{j-1}, k) < \beta < \alpha$. Then,

$$Pf = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sigma_0(\theta, \beta + i\eta)\tilde{f}_0(\beta + i\eta)$$
$$+ \sigma_1(\theta, \beta + i\eta)\tilde{f}_1(\beta + i\eta))\, \rho^{\beta + i\eta}\, d\eta + A + B + C.$$

A, B, and C are sums of residues from three types of poles, namely, poles of $\tilde{f}_0, \tilde{f}_1$, poles of σ_0, σ_1, and the coincidence of these. Let

$$N_0 = \{n \mid n \text{ is even, } c_0(M, n) = 0,\ 0 \leqslant n < \alpha\},$$
$$N_1 = \{n \mid n \text{ is odd, } c_1(M, n) = 0,\ 0 \leqslant n < \alpha\},$$
$$A = \sum_{\substack{n < \alpha \\ n \notin N_0 \cup N_1}} \left[\sigma_0(\theta, n)\, \rho^n \frac{1}{n!} f_0^{(n)}(0) + \sigma_1(\theta, n)\, \rho^n \frac{1}{n!} f_1^{(n)}(0)\right].$$

Note that $f_0^{(n)}(0) = 0$ when n is odd and $f_1^{(n)}(0) = 0$ when n is even.

$$B = \sum_{m=0}^{j/2} \frac{g_0(x, t; \nu_{2m})}{c_0'(M, \nu_{2m})}\tilde{f}_0(\nu_{2m})\, \rho^{\nu_{2m}}$$
$$+ \frac{g_1(x, t; \nu_{2m+1})}{c_1'(M, \nu_{2m+1})}\tilde{f}_1(\nu_{2m+1})\, \rho^{\nu_{2m+1}}, \qquad \nu_{2m} \notin N_0,\ \nu_{2m+1} \notin N_1.$$

Note that $\nu_{2m} \notin N_0$ and $\nu_{2m+1} \notin N_1$ guarantees that $\tilde{f}_0(\nu_{2m})$ and $\tilde{f}_1(\nu_{2m+1})$ are finite.

$$C = \sum_{n \in N_0} \frac{1}{n!} f_0^{(n)}(0) \frac{1}{c_0'(M, n)} (\rho^n (\log\rho)\ g_0(x, t; n) + g_0'(x, t; n)\, \rho^n)$$
$$+ \sum_{n \in N_1} \frac{1}{n!} f_1^{(n)}(0) \frac{1}{c_0'(M, n)} (\rho^n (\log\rho)\ g_1(x, t; n) + g_1'(x, t; n)\, \rho^n).$$

Here again $g_0'(x, t; \nu)$ denotes differentiation with respect to ν.

The restriction of $A+B+C$ to ∂D_M is the Taylor series of f up to order k. All other terms vanish on the boundary. Let

$$v(y)=f(y)-\sum_{n<\alpha} f^{(n)}(0)\, y^n.$$

Then

$$P(f)=\int_{-\infty}^{\infty} P_j(x,t;\, y)\, v(y)\, dy + A + B + C.$$

The term A is a global C^∞ function. Indeed, $\sigma_0(\theta,n)\,\rho^n$ for n even and $\sigma_1(\theta,n)\,\rho^n$ for n odd are polynomials in x and t. Each summand in B is exactly Γ_{ν_n} for appropriate n. Hence for regularity we must impose compatibility conditions

$$\tilde{f}_0(\nu_{2m})=\tilde{f}_1(\nu_{2m+1})=0 \qquad \text{for} \quad \nu_{2m}\notin N_0,\ \nu_{2m+1}\notin N_1,$$

ν_{2m}, $\nu_{2m+1}<\alpha$. These are global conditions on f. They imply $B=0$.

For term C to vanish we must require the local conditions $f_0^{(n)}(0)=0$, $n\in N_0$ and $f_1^{(n)}=0$, $n\in N_1$. We can rewrite this as $f^{(n)}(0)=0$, for $n\in N_0\cup N_1$.

It remains to verify the smoothness of $\int_{-\infty}^{\infty} P_j(x,t;\, y)\, v(y)\, dy$. Observe that $e^{-\alpha s}v(\pm e^s)$ belongs to $\Lambda_\alpha(\mathbb{R})$ as a function of s. Let $Q_{y_0}(s)$ be the Taylor polynomial of order k of $e^{-\alpha s}v(\pm e^s)$ at $\pm e^s=y_0$.

$$|v(y)-|y|^\alpha Q_{y_0}(\log|y|)|\leqq Cy^\alpha(\log|y/y_0|)^\alpha \qquad \text{for } y(\operatorname{sgn} y_0)>0.$$

Define $\psi(\nu)=\exp(-(\nu-\alpha)^{4N+2})$ for some large integer $N>\alpha$, $\psi(\beta+i\eta)$ is rapidly decreasing as $n\to\pm\infty$ and $1-\psi(\nu)$ vanishes to order $4N+2$ as ν approaches α. Write $P_j=P_j^{(1)}+P_j^{(2)}$, where

$$P_j^{(1)}(x,t;\, y)=\frac{1}{2\pi}\, y^{-1}\int_{-\infty}^{\infty} \tfrac{1}{2}(\sigma_0(\theta,\beta+i\eta)$$
$$+\sigma_1(\theta,\beta+i\eta))(\rho/y)^{\beta+i\eta}\psi(\beta+i\eta)\, d\eta \qquad (\beta\in I_j).$$

Because ψ is rapidly decreasing as $\eta\to\infty$, $P_j^{(1)}$ satisfies better estimates than P_j:

$$|X^\gamma P_j^{(1)}|\leqq C\rho^{-1-|\gamma|}/m_j(\rho/y).$$

$P_j^{(2)}$ satisfies the same estimates at P_j, but it is also orthogonal to $|y|^\alpha Q_{y_0}(\log|y|)$, provided degree $Q\leqq 4N+2$.

$$\left|X^\gamma\int_{-\infty}^{\infty} P_j^{(1)}(x,t;\, y)\, v(y)\, dy\right|\leqq C\int_{-\infty}^{\infty}[\rho^{-1-|\gamma|}/m_j(\rho/y)]\; y^\alpha\, dy$$
$$\leqq C\rho^{\alpha-|\gamma|}$$

because $\nu_{j-1}<\alpha<\nu_j$.

$$\left| X^{\gamma} \int_{-\infty}^{\infty} P_j^{(2)}(s, t; y)\, v(y)\, dy \right|$$

$$= \left| X^{\gamma} \int_{0}^{\infty} P_j^{(2)}(x, t; y)(v(y) - y^{\alpha} Q_{\rho}(\log y))\, dy \right.$$

$$\left. + X^{\gamma} \int_{-\infty}^{0} P_j^{(2)}(x, t; y)(v(y)) - |y|^{\alpha} Q_{-\rho}(\log |y|))\, dy \right|$$

$$\leqq Cd((x, t), \partial D_M)^{\alpha - |\gamma|} \qquad \text{for } |\gamma| > \alpha.$$

In all, $|X^{\gamma} P_j(v)(x, t)| \leqq Cd((x, t), \partial D_M)^{\alpha - |\gamma|}$ for all $|\gamma| > \alpha$. It follows easily that $P_j(v)$ can be extended to a function that is Γ_{α} in a neighborhood of $(0, 0)$. (For a more complicated extension of this kind see I.4.) This concludes the proof.

The simplest case of Theorem (2.9) is the case $M = 0$. In that case,

$$v_{2k} = 4k + 2,\ v_{2k+1} = 4k + 3 \qquad k = 0, 1, 2, \ldots$$

(see [2, p. 104 (50) and (51)]). The compatibility conditions are thus all local.

As was first observed by H. Hung (personal communication) the local compatibility conditions can be found by differentiating the equation with respect to x. In this particular case the local conditions suffice for regularity. A similar phenomenon for the heat operator is treated in [13]. It seems likely that the missing global compatibility in the case of the heat operator could be explained by means of the Mellin transform. The special functions that arise in that case are Laguerre functions.

A great deal is known about the behavior of the values v_j as M varies (see [18]). In particular, it is easy to see that as M tends to infinity so does v_0 and as M tends to $-\infty$, v_0 tends to 0. Note that the Poisson kernel P_0 vanishes to order v_0, exactly, at the characteristic point. This phenomenon was observed by Gaveau [5, p. 106] for $\mathscr{L}_0$ in the Heisenberg ball $|z|^4 + t^2 < 1$. (See Section 3 or I.1 for notations.) In that case we can explain why the Poisson kernel vanishes to order 2 by the function $u(z, t) = t$, which is homogeneous in the appropriate sense of order 2 and is the analogue in the context of $\mathscr{L}_0$ of $g_0(x, t; v_0)\, \rho^{v_0}$ (for $(v_0 = 2)$.

For a general domain D we will prove a theorem that is far more crude. Suppose that m is a characteristic point of ∂D. Translate in t so that $m = (0, 0)$ and let $D_M = \{(x, t) \mid t > Mx^2\}$ be tangent to D at $(0, 0)$ to third order. The critical index for D at m is defined to be v_0, the first positive pole of the symbol of the Poisson kernel for D_M.

(2.10) THEOREM. *For any* $f \in C(\partial D)$, *there is a unique function* $u \in C(\bar{D})$ *such that* $Lu = 0$ *in* D *and* $u = f$ *on* ∂D. *Let* ϕ_1, ϕ_2 *belong to* $C_0^\infty(\mathbb{R}^2)$ *be supported in a small neighborhood of* m *with* $\operatorname{supp} \phi_2 \subset \{\phi_1 = 1\}$. *If* $\phi_1 f \in \Lambda_\alpha(\partial D)$ *and* $\alpha < v_0$, *then* $\phi_2 u \in \Gamma_\alpha(\bar{D})$. *If* $\alpha \geqslant v_0$, *then for some* $f \in \Lambda_\alpha(\partial D)$, $\phi_2 u \notin \Gamma_\alpha(\bar{D})$.

Proof. The existence of a continuous solution is guaranteed by the construction of barriers (see Lemma (3.2), [1, 10]). The solution is unique by the weak maximum principle. For higher regularity at characteristic points we use another kind or argument.

We may suppose that $m = (0, 0)$ and ∂D by $t = Mx^2 + g(x)$ with $g(x) = O(x^3)$. Assign local coordinates to ∂D by projection $\phi \colon (-\varepsilon, \varepsilon) \to \partial D$; $\phi(x) = (x, Mx^2 + g(x))$. Denote $u_{2k}(x, t) = g_0)x, t; 2k) \rho^{2k}$

$$u_{2k+1}(x, t) = g_1(x, t; 2k+1) \rho^{2k+1}$$

$u_n(x, t)$ is a polynomial in x and t. Also $u_n(x, t)$ does not vanish on $t = Mx^2$ provided $j < v_0$. Assume $\alpha < v_0$ and α is not an integer. It follows by an inductive procedure that for any $\phi_1 f \in \Lambda_\alpha(\partial D)$, we can write

$$\begin{aligned} &g = f - \textstyle\sum_{n<\alpha} c_n u_n|_{\partial D}, \qquad \text{where } c_n \text{ depends only} \\ &\text{on } f^{(j)}(0), j \leqslant n, \text{ and } g \text{ satisfies} \\ &e^{-\alpha s} g(\phi(\pm e^s)) \in \Lambda_\alpha(\mathbb{R}), s < \log \varepsilon, \end{aligned} \tag{2.11}$$

as a function of s. Since u differs from the Poisson extension of g by a smooth function, we may as well assume $f^{(n)}(0) = 0$, $n < \alpha$ and $g = f$.

Define

$$\begin{aligned} B_1 &= \{(x, t) \mid t > Mx^2 + 1, (x^4 + 4t^2)^{1/4} < 10\}, \\ B_2 &= \{(x, t) \mid t > Mx^2 + 1/2, (x^4 + 4t^2)^{1/4} < 20\}, \\ V_1 &= \{(x, t) \mid d((x, t), (1, M)) < 1/2\}, \\ V_2 &= \{(x, t) \mid d((x, t), (1, M) < 3/4\}. \end{aligned}$$

For a subset $S \subset R^2$, define $\delta S = \{(\delta x, \delta^2 t) \mid (x, t) \mid (x, t) \in S\}$. Let $I_\delta = \{s \mid \phi(s) \in \delta V_1\}$. For small δ, $\delta B_2 \subset D$. If $Lu = 0$ in D, then

$$\|X^\gamma u\|_{L^\infty(\delta B_1)} \leqslant C \delta^{-|\gamma|} \|u\|_{L^\infty(\delta B_2)} \tag{2.12}$$

$$\begin{aligned} \|X^\gamma u\|_{L^\infty(\delta V_1 \cap D)} \leqslant C \varepsilon^{-|\gamma|} (\|u\|_{L^\infty(\delta V_2 \cap D)} \\ + \|f(\phi(e^s))\|_{\Lambda_\alpha(I_\delta)}), \end{aligned} \tag{2.13}$$

where $\varepsilon = d((x, t), \partial D)$. There is a similar estimate to (2.13) replacing x with $-x$.

These estimates are just dilations of well-known estimates for $\delta = 1$. The first amounts to interior regularity for L. The second is the dilation of a well-known Lipschitz boundary estimate for solutions to uniformly elliptic equations.

Denote $B = \{(x, t) \mid (x^4 + 4t^2)^{1/4} < c\}$. For some small fixed value of c, $g_0(x, t; \alpha)\rho^\alpha$ is bounded from below on $D \cap \delta B$. Also, from (2.11), $f = O(\rho^\alpha)$ on δD. The maximum principle applied to the region $D \cap B$ implies $|u(x, t)| \leqq C\, |g_0(x, t; \alpha)\rho^\alpha| \leqslant C\rho^\alpha$ for $(x, t) \in D \cap B$. Hence,

$$\|u\|_{L^\infty(\delta B_2)} + \|u\|_{L^\infty(\delta V_2 \cap D)} \leqq C\delta^\alpha.$$

Estimate (2.11) says

$$\|f(\phi(e^s))\|_{\Lambda_\alpha(I_\delta)} \leqq C\delta^\alpha.$$

Therefore,

$$|X^\gamma(\phi_2 u)| \leqslant C(\varepsilon^{-|\gamma|} + \rho^{-|\gamma|})\,\rho^\alpha \leqq C\varepsilon^{\alpha - |\gamma|},$$

using (2.12) and (2.13). As noted in Theorem (2.4), this estimate implies $\phi_2 u \in \Gamma_\alpha(\bar{D})$.

For failure of regularity, $\alpha \geqslant v_0$, consider $u_1(x, t) = g_0(x, t; v_0)\rho^{v_0}$ and $u_2(x, t) = (\partial/\partial v)[g_0(x, t; v)\rho^v]|_{v = v_0}$.

Because ∂D is tangent to ∂D_M to third order and because $g_0(x, t; v_0)$ vanishes on ∂D_M, $u_1(x, t) = O(\rho^{v_0 + 1})$ on D. However, $u_1(0, t) = ct^{v_0/2}$, $c \neq 0$. Thus $u_1 \in \Lambda_{v_0 + 1}(\partial D)$, but $u_1 \notin \Gamma_\beta(\bar{D})$ for $\beta > v_0$, unless v_0 is an even integer. A similar analysis shows $u_2 \in \Lambda_{v_0}(\partial D)$, but $u_2 \notin \Gamma_{v_0}(\bar{D})$. The singularity along the t axis is $t^{v_0/2}\log t$.

We conclude this section with several remarks on what is to come. Sections 3, 4, and 5 are devoted to the proof of an analogue of Theorem (2.10) for the operator $\mathscr{L}_0$ (Theorem (5.2)). The analogue of (2.11) is the space Γ_β^0 described in Section 4. The barriers corresponding to $g_0(x, t; v_0)\rho^{v_0}$ are constructed in Theorem (5.1). Part I concerns the analogue of (2.13). In this context, we would like to make explicit the connection between the proofs of Theorems (2.9) and (2.10), hidden in estimate (2.13). In order to prove (2.13), one has to make estimates on the Poisson kernel for an elliptic operator. The simplest way to do this is the calculate the symbol of the Poisson kernel and use pseudo-differential operators. This is exactly what was done explicitly for the symbol of P_j in Theorem (2.9). In both theorems, non-isotropic dilations of the estimates give the full picture once the elliptic problem is understood. The non-characteristic part of the Dirichlet problem for $\mathscr{L}_0$ is not elliptic, which is why it requires the analysis carried out in Part I.

3. Preliminary Results on the Heisenberg Group

We begin in this section with a relatively weak result for $\mathscr{L}_0$ that does not require the heavy machinery of Part I. It is valid near any singular point not just at strongly isolated characteristic points (see Sect. 4 for the definition). At the same time, we will introduce some of the barriers analogous to $g_0(x, t; \nu_0)\,\rho^{\nu_0}$ of Section 2.

Denote the Heisenberg group by $\mathbb{H}^n$. The underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. Group multiplication is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\,\mathrm{Im}\, z \cdot \bar{z}'),$$

where

$$z = x + iy \in C^n, \qquad t \in \mathbb{R},\ z \cdot \bar{z}' = \sum_{j=1}^{n} z_j \bar{z}'_j.$$

A basis for the left-invariant vector fields on $\mathbb{H}^n$ is $X_j = \partial/\partial x_j + 2y_j\,\partial/\partial t$, $Y_j = \partial/\partial y_j - 2x_j\,\partial/\partial t$, $j = 1,\dots, n$, and $T = \partial/\partial t$. The commutation relations are $[X_j, Y_k] = -4\delta_{jk} T$ and all other commutators vanish. The natural norm on $\mathbb{H}^n$ is $|(z, t)| = (|z|^4 + t^2)^{1/4}$. From now on we will use the letters x, and y (and w) to denote elements of $\mathbb{H}^n$, not $\mathbb{R}^n$. A left-invariant distance function is given by $d(x, y) = |x^{-1}y|$, where x and y belong to $\mathbb{H}^n$. Note that $d(x, y) = d(y, x)$. There is an approximate triangle inequality $d(x, y) \leqslant C(d(x, w) + d(w, y))$ (see [4]). If $0 < \beta < 1$, Γ_β is the space of functions f on $\mathbb{H}^n$ such that for any x and y in $\mathbb{H}^n$,

$$|f(x) - f(y)| < Cd(x, y)^\beta$$

(see Section 4). For now we will only be concerned with small values of β, $\beta < 1$. For a smooth bounded domain D or a hypersurface S in $\mathbb{H}^n$ define $\Gamma_\beta(\bar{D})$ and $\Gamma_\beta(S)$ as the restriction of Γ_β to these subsets.

The operator (see introduction)

$$\mathscr{L}_0 = -\frac{1}{4} \sum_{j=1}^{n} X_j^2 + Y_j^2$$

is a left-invariant operator on $\mathbb{H}^n$, homogeneous of degree 2 with respect to the natural dilations $\delta(z, t) = (\delta z, \delta^2 t)$. A singular point x_0 of ∂D is a point at which X_j and Y_j are tangent to ∂D for all j. Making a left translation to the origin, the equation of ∂D at the origin is

$$t = q(z) + g(z), \qquad g(z, t) = O(|(z, t)|^3).$$

$q(z)$ is a quadratic form in $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$.

Let us examine first the simplest case, namely, a rotation and dilation invariant region $D_M = \{(z,t) \mid t > M|z|^2\}$. Denote $\rho = (|z|^4 + t^2)^{1/4} = |(z,t)|$; $\tau = t/\rho^2$. Then

$$\mathscr{L}_0(u(\tau)\rho^\lambda) = 4(1-\tau^2)^{1/2}(L_\lambda u)(\tau)\rho^{\lambda-2},$$

where

$$L_\lambda = (1-\tau^2)\frac{d^2}{d\tau^2} - (n+1)\tau\frac{d}{d\tau} + \frac{1}{4}\lambda(\lambda+2n).$$

This is a Jacobi operator. In fact, when $n=1$ it is just the classical Legendre operator. One solution is denoted $P^{n/2}_{\lambda/2}(\tau)$ [18, p. 80]. We are interested in the solution that is smooth as a function of (z,t) as $\tau \to 1$. It is given by

$$g_\lambda(\tau) = F\left(-\lambda/2, n+\lambda/2; \frac{n+1}{2}; \frac{1}{2}(1-\tau)\right).$$

When $0 < \lambda < 2$, one can see that $g_\lambda(1) = 1$ and $g_\lambda(\tau) \to -\infty$ as $\tau \to -1^+$. Therefore g_λ has a zero τ_λ. One can check [2, p. 110 (14)] that as $\lambda \to 0^+$, $\tau_\lambda \to -1^+$. The function $g_\lambda(\tau)\rho^\lambda$ is an example of a function that vanishes on ∂D_M for $M = \tau_\lambda/(1-\tau_\lambda^2)^{1/2}$. Yet, as $\lambda \to 0$, $g_\lambda(\tau)\rho^\lambda$ belongs (locally) to $\Gamma_\lambda(\bar{D}_M)$ and no better smoothness class.

(3.1) THEOREM. *Let D be a smooth bounded domain in $\mathbb{H}^n$. The Dirichlet problem $\mathscr{L}_0 u = 0$ in D, $u|_{\partial D} = f$ has a unique solution $u \in C(\bar{D})$ for all $f \in C(\partial D)$. There is a positive β_0 depending on D such that for all $\beta < \beta_0$, $f \in \Gamma_\beta(\partial D)$ implies $u \in \Gamma_\beta(\bar{D})$. For any $\beta > 0$ there is a domain D and $f \in C^\infty(\partial D)$ such that the solution u to the Dirichlet problem with boundary values f does not belong to $\Gamma_\beta(\bar{D})$.*

Proof. The existence of a solution u for continuous boundary values can be obtained by the Perron method. The continuity of u up to the boundary at non-characteristic points was proved by Bony [1], who did it by constructing barriers. Gaveau [5] proved that u is continuous at every boundary point in the case of the Heisenberg ball $|z|^4 + t^2 < 1$ by probabalistic means. His method extends to any smooth domain. The solution $u \in C(\bar{D})$ is unique by the weak maximum principle. We give an explicit construction of barriers (at any boundary points) below. This gives an independent proof that $u \in C(\bar{D})$, although we will not carry out the well-known argument [10]. Examples of domains for which solutions to the Dirichlet problem are not smooth are given by smooth truncations of D_M above (for large negative values of M) and by $u = g_\lambda(\tau)\rho^\lambda$ for appropriate λ.

Let $S(1) = \{(z,t) \mid |z|^4 + t^2 = 1\}$. A cone is given by $T_\varepsilon(\omega) = \{(\delta z, \delta^2 t) \mid 0 < \delta < \varepsilon,\ (z,t) \in \omega\}$, where ω is a *spherical cap* in $S(1)$. By spherical cap we mean the intersection of $S(1)$ with a small Euclidean ball. The main point of the Γ_β estimate for u is that smooth domains D satisfy the analogue of the exterior cone condition. Because ∂D has bounded second derivatives, for any $(z_0, t_0) \in \partial D$, there exists $\omega \subset S(1)$ such that diameter $(\omega) > \varepsilon$ and $\{(z_0, t_0) \cdot (z, t) \mid (z, t) \in T_\epsilon(\omega)\} \subset {}^cD$.

We will now construct barriers in the complement of these exterior cones. Let θ be a smooth coordinate for the sphere $S(1)$.

(3.2) LEMMA. *For any $\beta_0 > 0$ and for any spherical cap ω in $S(1)$, there exists a subset $\Omega \subset S(1)$ such that $S(1)/\Omega \subset \omega$ and a barrier function $g_\theta(\theta)\rho^\beta$ satisfying $0 < \beta < \beta_0$ and*

$$\text{(a)} \quad \mathscr{L}_0 g_0(\theta)\rho^\beta = 0, \qquad \theta \in \Omega,\ \rho > 0,$$

$$\text{(b)} \quad 0 < c_1 < g_0(\theta) < c_2 \qquad \text{for} \quad \theta \in S(1)/\omega.$$

Proof. The volume element for $\mathbb{C}^n \times \mathbb{R}$ can be rewritten $\rho^{2n+1} d\rho\, d\sigma(\theta)$, where $d\sigma(\theta)$ is a smooth non-vanishing measure on $S(1)$. The power $2n+1$ appears because the "homogeneous" dimension of $\mathbb{C}^n \times \mathbb{R}$ is $2n+2$: the t direction counts twice. Because X_j and Y_j have degree -1 with respect to the non-isotropic dilations, $X_j = \rho^{-1} X_j^\theta + a_j(\theta)\,\partial/\partial\rho$, $Y_j = \rho^{-1} Y_j^\theta + b_j(\theta)\,\partial/\partial\rho$, where X_j^θ and Y_j^θ are vector fields in θ only. Hence,

$$X_j(g(\theta)\rho^\lambda) = (X_{j\lambda} g)\rho^{\lambda-1},$$

$$Y_j(g(\theta)\rho^\lambda) = (Y_{j\lambda} g)\rho^{\lambda-1},$$

where $X_{j\lambda} = X_j^\theta + \lambda a_j$, $Y_{j\lambda} = Y_j^\theta + \lambda b_j$.

Define L_λ by the equation

$$\mathscr{L}_0(g(\theta)\rho^\lambda) = (L_\lambda g)\rho^{\lambda-2};$$

then

$$L_\lambda = \sum_j X_{j(\lambda-1)} X_{j\lambda} + Y_{j(\lambda-1)} Y_{j\lambda}.$$

Denote

$$\langle F, G\rangle = \iint F(\rho, \theta)\,\overline{G(\rho, \theta)}\,\rho^{2n+1}\, d\rho\, d\sigma(\theta)$$

and

$$(f, g) = \int f(\theta)\,\overline{g(\theta)}\, d\sigma(\theta).$$

Choose $\psi \in C_0^\infty(0, \infty)$. X_j and Y_j are self-adjoint up to sign on $\mathbb{C}^n \times \mathbb{R}$. In other words,

$$\langle X_j(f(\theta)\,\rho^\lambda),\ g(\theta)\,\psi(\rho)\rangle + \langle f(\theta)\,\rho^\lambda, X_j(\,g(\theta)\,\psi(\rho))\rangle = 0.$$

Consequently,

$$\{(X_{j\lambda} f,\ g) - (f, X_{j(-\lambda-(2n+1))}\,g)\} \int_0^\infty \rho^{\lambda+2n}\psi(\rho)\,d\rho = 0.$$

Hence,

$$(X_{j\lambda} f,\ g) = (f, -X_{j(-\lambda-(2n+1))}\,g).$$

Similarly,

$$(Y_{j\lambda} f,\ g) = (f, -Y_{j(-\lambda-(2n+1))}\,g).$$

Denote

$$D(\phi) = \sum ((X_j^\theta - na_j)\phi, X_j^\theta - na_j)\phi) + ((Y_j^\theta - nb_j)\phi, (Y_j^\theta - nb_j)\phi).$$
$$h(\theta) = \sum a_j^2 + b_j^2; \qquad H(\phi) = (\phi, h\phi).$$

The adjoint formulas for $X_{j\lambda}$ and $Y_{j\lambda}$ show that

$$(L_\lambda\phi, \phi) = -D(\phi) + (\lambda + n)^2 H(\phi). \tag{3.3}$$

Also, since $a_j = X_j\rho$ and $b_j = Y_j\rho$,

$$h = |z|^2/\rho^2. \tag{3.4}$$

Next, we prove the subcoercive (or subelliptic) estimate

$$D(\phi) \geqslant c_1\,\|\phi\|^2_{L^2_{1/2}} - c_2 H(\phi), \qquad \text{for all } \phi \in C_0^\infty(\Omega)\ \ \Omega \subset S(1). \tag{3.5}$$

Because a_j and b_j vanish to order 1 in z and h vanishes to order 2 (see (3.4)),

$$D(\phi) + CH(\phi) \geqslant \tfrac{1}{2} \sum (X_j^\theta\phi, X_j^\theta\phi) + (Y_j^\theta\phi, Y_j^\theta\phi).$$

It is easy to check that X_j^θ and Y_j^θ on $S(1)$ inherit from the ambient space $\mathbb{H}^n$ the property that X_j^θ, Y_j^θ, $[X_j^\theta, Y_j^\theta]$ span the tangent space at each point. (In fact, L_λ degenerates only on the equator $S(1) \cap \{t = 0\}$.) Therefore, by a theorem of Kohn [11]; see also (1.1))

$$\sum (X_j^\theta, X_j^\theta\phi) + (Y_j^\theta\phi, Y_j^\theta\phi) + C\,\|\phi\|^2_{L^2} \geqslant \|\phi\|^2_{H_{1/2}(\Omega)}$$

for $\phi \in C_0^\infty((\Omega))$. Writing ϕ as the integral of its gradient it is easy to see that $CD(\phi) + CH(\phi) \geqslant \|\phi\|_{L^2}^2$ and (3.5) follows.

Estimate (3.5) implies that the inclusion mapping from the closure of $C_0^\infty(\Omega)$ in the norm $D(\phi) + CH(\phi)$ to the closure of $C_0^\infty(\Omega)$ in the norm $H(\phi)$ is compact. The Lax–Milgram lemma and spectral decomposition for self-adjoint compact operators leads to the existence of eigenfunctions $g_j(\theta)$ vanishing on $\partial\Omega$ with eigenvalues μ_j such that

$$D(g_j, v) = \mu_j H(g_j, v) \qquad \text{for all} \quad v \in C_0^\infty(\Omega).$$

$D(\cdot, \cdot)$ and $H(\cdot, \cdot)$ are the polarizations of $D(\cdot)$ and $H(\cdot)$. Let $(\lambda_j + n)^2 = \mu_j$, then by (3.3), $L_{\lambda_j} g_j = 0$, or

$$\mathscr{L}_0(g_j(\theta)\rho^{\lambda_j}) = 0.$$

In particular, the first eigenvalue is given by

$$(\lambda_0 + n)^2 = \inf \left\{ \frac{D(\phi)}{H(\phi)} \;\middle|\; \phi \in C_0^\infty(\Omega) \right\}.$$

By standard variational arguments the corresponding eigenfunction $g_0(\theta) \geqslant 0$ and $(\lambda_0 + n)^2$ decreases continuously as the region Ω increases. In particular, when $\Omega = S(1)$, the function $g_0(\theta) \equiv 1$ and $D(g_0)/H(g_0) = n^2$. Therefore, for smaller regions $(\lambda_0 + n)^2 > n^2$, i.e., $\lambda_0 > 0$. The function of Lemma (3.2) is defined as $g_0(\theta)\rho^{\lambda_0}$ with $\beta = \lambda_0$, and Ω a region sufficiently large that $S(1)/\Omega \subset\subset \omega$.

It remains to prove that Ω can be chosen so that $\beta < \beta_0$ and so that $g_0(\theta)\rho^\beta$ satisfies (b). First we will show that as $S(1)/\Omega$ shrinks to a point, λ_0 tends to zero. The case of a spherical cap shrinking to the points when $|z| = 0$ was already checked explicitly for rotation invariant regions. Away from $|z| = 0$, the weight h in the norm $H(\phi)$ does not vanish. Moreover,

$$D(\phi) \leqslant n^2 H(\phi) + C\left(\int |\nabla\phi|^2 + \left(\int |\nabla\phi|^2\right)^{1/2} H(\phi)^{1/2}\right).$$

In order to show $\lambda_0 \to 0$, it therefore suffices to find $\phi \in C_0^\infty(\Omega)$ such that $\int |\nabla\phi|^2/H(\phi) \to 0$ as 0 shrinks to a point other than $|z| = 0$.

This is easy on spheres of dimension $k \geqslant 3$. Suppose the complement of Ω has radius δ. The problem reduces to an estimation of the usual Dirichlet integral of a function that vanishes in a small disc. If $f(r)$ is the function satisfying $f(\delta) = 0$ and

$$\begin{aligned} f'(r) &= 0 & r &< \delta \\ &= \delta^{-1} & \delta &< r < 2\delta \\ &= 0 & r &> 2\delta, \end{aligned}$$

then $f(r) = 1$ for $r \geqslant 2\delta$. If $\phi(x) = f(|x|)$, $x \in \mathbb{R}^k$, then

$$\frac{\int_{|x|<1} |\nabla \phi(x)|^2 \, dx}{\int_{|x|<1} |\phi(x)|^2 \, dx} \simeq \int_0^1 f'(r)^2 r^{k-1} \, dr \leqq C\delta^{k-2}.$$

When $k = 2$, we need a slightly different function:

$$\begin{aligned} f'(r) &= 0 & & r < \delta \\ &= r^{-1} \left(\log \frac{1}{\delta}\right)^{-1} & & \delta < r < \delta^{1/2} \\ &= 0 & & r > \delta^{1/2}. \end{aligned}$$

Then $f(r) > 1/10$ for $r > \delta^{1/2}$ and

$$\frac{\int_0^1 f'(r)^2 r \, dr}{\int_{\frac{1}{2}}^1 f(r)^2 r \, dr} \leqq C \left(\log \frac{1}{\delta}\right)^{-1}.$$

Replacing Ω by a larger region, we can assume that $\partial\Omega$ is non-characteristic at every point relative to the operator L_λ. The theorem of Kohn and Nirenberg [12, Theorem 3] then implies that $g_0(\theta) \in C^\infty(\bar{\Omega})$.

Bony's maximum principle [1] says (in the case of $\mathscr{L}_0$) that if $u \in C(\bar{D}) \cap C^\infty(D)$, u real, $\mathscr{L}_0 u = 0$, and u attains its maximum in D, then u is constant. Since $\mathscr{L}_0(g_0(g)\rho^{\lambda_0}) = 0$ and $g_0(\theta)\rho^{\lambda_0} \geqslant 0$ for $\theta \in \Omega$, Bony's maximum principle implies $g_0(\theta) > 0$ for $\theta \in \Omega$. Therefore, on a slightly smaller region $\omega' \subset\subset \omega$, $0 < c_1 < g_0(\theta) < c_2$, $\theta \in \omega'$. This concludes the proof of Lemma (3.2).

We can now prove Theorem (3.1). Suppose $f \in \Gamma_\beta(\partial D)$ with β small. Fix $(z_0, t_0) \in \partial D$ and compare $u(z, t) - u(z_0, t_0)$ to the left translation by (z_0, t_0) of a barrier function $g_0(\theta)\rho^\beta$ in the region $V = \{(z, t) \mid d((z, t), (z_0, t_0)) < \varepsilon\} \cap D$. It follows from (3.2b) that the left translate of $g_0(\theta)\rho^\beta$ is bounded above and below by multiples of $d((z, t), (z_0, t_0))^\beta$, for each β, there is a finite collection of regions $\omega' \subset\subset \omega$ and associated barrier functions $g_0(\theta)\rho^\beta$ such that for every point of ∂D at least one barrier functions is defined and bounded above and below by multiples of $d((z, t), (z_0, t_0))^\beta$ in V. Thus the multiples, while they may depend on β, are uniform for all boundary points. It follows from the (weak) maximum principle that

$$|u(z, t) - u(z_0, t_0)| \leqq C_\beta d((z, t), (z_0, t_0))^\beta \, \|f\|_{\Gamma_\beta(\partial D)}. \tag{3.6}$$

Let X^γ denote any monomial of length $|\gamma|$ in X_j and Y_j. Let

$$\begin{aligned} B_1 &= \{(z, t) \mid |(z, t)| < 1\}, \\ B_2 &= \{(z, t) \mid |(z, t)| < 2\}, \\ \delta B &= \{(\delta z, \delta^2 t) \mid (z, t) \in B\}. \end{aligned}$$

A dilation of Kohn's interior estimate for solutions to $\mathscr{L}_0 u = 0$ gives

$$\|X^\gamma u\|_{L^\infty(\delta B_1)} \leqq C_\gamma \delta^{-|\gamma|} \|u\|_{L^\infty(\delta B_2)}. \tag{3.7}$$

Estimate (3.6) and left translations to small balls near ∂D of estimate (3.7) give

$$|X^\gamma u(z,t)| \leqq Cd((z,t), \partial D)^{\beta - |\gamma|} \|f\|_{\Gamma_\beta(\partial D)}. \tag{3.8}$$

This implies $u \in \Gamma_\beta(\bar{D})$.

Remark. If V is a small neighborhood of a non-characteristic point, then estimate (3.8) is valid for any $\beta < 1$.

Indeed, if $r(z,t)$ is a defining function for ∂D and $(0,0) \in \partial D$ is a non-characteristic point, the dilation of ∂D is

$$\delta^{-1}(\partial D) = \{(z,t) \mid r(\delta z, \delta^2 t) = 0\}.$$

If $r(z,t) = a \cdot x + b \cdot y + ct +$ higher order terms, then the limit as $\delta \to 0$ of $\sigma^{-1}(\partial D)$ is the plane $a \cdot x + b \cdot y = 0$. The function $a \cdot x + b \cdot y$ has homogeneous degree 1 and satisfies $\mathscr{L}_0(a \cdot x + b \cdot y) = 0$. It is a barrier function at $(0,0)$ for the region $\{(z,t) \mid a \cdot x + b \cdot y > 0\}$ and ∂D is arbitrarily close to this region under dilation.

This remark is important for the following reason. We do not known yet if the solution to the Dirichlet problem constructed in this way coincides with the L^2 construction mentioned in the introduction and used in Part I. Suppose that $f \in C^\infty(\partial D)$ and u is the solution to the Dirichlet problem (3.1) with boundary values f. Let V be a small neighborhood of a non-characteristic point, then (3.8) for $\beta > 1/2$ shows that $X_j u$ and $Y_j u$ belong to $L^2(V \cap D)$. This proves that u is the solution to the Dirichlet problem in $V \cap D$ in the L^2 sense. Therefore, the regularity theorem of Kohn and Nirenberg applies to u, and we have proved

(3.9) PROPOSITION. *If $f \in C^\infty(\partial D)$ and u is the solution to $\mathscr{L}_0 u = 0$, $u/\partial D = f$ given by the Perron process then u is C^∞ up to the boundary at all non-characteristic points of ∂D.*

4. Γ_β SPACES AND STRONGLY ISOLATED CHARACTERISTIC POINTS

Recall that the space Γ_β for arbitrary β, $0 < \beta < \infty$, is defined as the class of continuous functions f such that for any $x \in \mathbb{H}^n$ and any $\delta > 0$, there exists a polynomial in $y \in \mathbb{H}^n$ denoted $P_{(x,\delta)}(y)$ such that

$$|f(y) - P_{(x,\delta)}(y)| \leqslant C\delta^\beta \qquad \text{whenever } d(x,y) < \delta.$$

(See [4, 6] and I.4.)

In Part I we discussed the restriction of Γ_β to a hypersurface M that was non-characteristic with respect to $\mathscr{L}_0$. We will now consider a hypersurface S that contains a characteristic point and describe the restriction of Γ_β to S, which we denote $\Gamma_\beta(S)$. Because we are only concerned with local properties, we will always ignore the boundary of S.

Let r be a defining function for S. A point $x \in S$ is characteristic if $\sum_{j=1}^n X_j r(x)^2 + Y_j r(x)^2 = 0$. We say that the characteristic point x is *strongly isolated* if for some $c > 0$,

$$\sum_{j=1}^{n} X_j r(y)^2 + Y_j r(y)^2 \geqslant c\,|x^{-1}y|^2 \qquad \text{for} \quad y \in S.$$

Make a left translation so that u is sent to the origin. To say that S is characteristic at 0 means that r can be written in the form

$$r(z, t) = t - q(z) + R(z, t),$$

where $q(z) = \sum_{i,j=1}^{2n} a_{ij} x_i x_j$ $(z_j = x_j + ix_{j+n}$ and $R(z, t) = O(|(z, t)|^3)$. Let $r_0(z, t) = t - q(z)$ and $S_0 = \{(z, t) \mid r_0(z, t) = 0\}$. It is easy to check that 0 is a strongly isolated characteristic point of S if and only if 0 is an isolated characteristic point of S_0.

EXAMPLE. In $\mathbb{H}^1$, 0 fails to be an isolated characteristic point of S_0 if and only if

$$r_0(z, t) = t - ax_1^2 + a^{-1}x_2^2 \qquad (z = x_1 + ix_2).$$

Thus characteristic points that are not strongly isolated are special kinds of saddle points.

We will now define a "neighborhood system" in the neighborhood of a strongly isolated singular point x on S. This system will be a countable collection of open balls that are either disjoint from S or intersect S in a non-characteristic piece M. For simplicity we will assume that $x = 0$. Denote the dilation by 2^j by

$$R_j(z, t) = (2^j z, 2^{2j} t).$$

Consider the annular regions

$$A_j = \{(z, t) \mid 2^{-j-1} < |(z, t)| < 2^{-j+1}\}.$$

If r is the defining function for S, denote

$$r_j(z, t) = 2^{2j} r \circ R_{-j}.$$

The zero set of r_j is

$$S_j = \{(z, t) \mid r_j(z, t) = 0\} = R_j(S).$$

Note that $A_0 \cap S_j = R_j(A_j \cap S)$. We can partition A_0 into a finite collection of overlapping open balls G_i^j, V_i^j such that

(i) $d(G_i^j, S_j) > c$,

(ii) V_i^j is a small neighborhood of a non-characteristic point on $A_0 \cap S_j$ as we have dealt with in Part *I*.

This can be accomplished because the characteristic point in question is isolated. Thus for sufficiently large j, $A_0 \cap S_j$ contains no characteristic points.

Furthermore, because the low order terms in the Taylor series for r are $t - q(z)$,

$$\left| \frac{\partial^{\gamma_1}}{\partial z_\gamma^{\ 1}} \frac{\partial^{\gamma_2}}{dt^{\gamma_2}} r_j(z, t) \right| \leqslant C_{\gamma_1, \gamma_2} \qquad \text{for} \quad (z, t) \in A_0 \tag{4.1}$$

uniformly in j. Finally, because 0 is a strongly isolated singular point,

$$\sum_{i=1}^{n} (X_i r_j(y))^2 + (Y_i r_j(y))^2 \geqslant c, \qquad y \in A_0, \tag{4.2}$$

independent of j.

Estimates (4.1) and (4.2) guarantee that the bounds involved in the construction of special coordinates in Part I, Section 2 are independent of j. It follows that the estimates on Poisson-type extension operators given in Part I, Section 4 are valid in neighborhoods V_i^j independent of i and j.

Denote by $\Gamma_\beta^0(S)$ the class of functions f on S such that $2^{j\beta} f \circ R_{-j}$ belongs to $\Gamma_\beta(A_0 \cap S_j)$ uniformly in j. (Recall that $A_0 \cap S_j$ is a non-characteristic hypersurface, so that the restriction of Γ_β to $A_0 \cap S_j$ is described by Theorem I.4.3.)

(4.3) THEOREM. *Suppose that $\beta > 0$ and β is not an integer. Every function in $\Gamma_\beta(S)$ can be written as the sum of a function in $\Gamma_\beta^0(S)$ and the restriction to S of a polynomial. Conversely, every such sum belongs to $\Gamma_\beta(S)$.*

Proof. Suppose that $f(x)$ belongs to Γ_β and $k < \beta < k + 1$. The Taylor polynomial of f at 0 of homogeneous degree k is given by

$$P(x) = \sum_{|\gamma| \leqslant k} \frac{1}{\gamma!} D^\gamma f(0)\, x^\gamma,$$

where D^γ is the left-invariant extension of the differential operator $\partial^\gamma/\partial y^\gamma$ at the origin. Let $g(x) = f(x) - P(x)$. By the proposition of Part I, Appendix A,

$$|D^\gamma g(x)| \leqslant C\,|x|^{\beta-|\gamma|} \qquad \text{for} \quad |\gamma| \leqslant k. \tag{4.4}$$

We intend to prove that g belongs to $\Gamma^0_\beta(S)$. Denote the Taylor polynomial of homogeneous degree k for g at u by

$$P_x(y) = \sum_{|\gamma| \leqslant k} \frac{1}{\gamma!} D^\gamma g(x)(x^{-1}y)^\gamma.$$

Estimate (4.4) implies that $2^{j\beta} P_x \circ R_{-j}(y)$ has C^{2k} norm bounded uniformly in j for $y \in A_0 \cap S_j$.

Take a Taylor polynomial $Q^j_x(\,)$ for $2^{j\beta} P_x \circ R_{-j}$ at $R_j(x)$ in local coordinates on $A_0 \cap S_j$. Then as in I Appendix A,

$$|2^{j\beta} g \circ R_{-j}(y) - Q^j_x(y)| \leqslant C\delta^\beta \qquad \text{whenever } d(y, R_j(x)) < \delta.$$

Thus by the characteristic of Theorem I.4.3, $2^{j\beta} g \circ R_{-j}$ belongs to $\Gamma_\beta(A_0 \cap S_j)$ with bounds independent of j. Consequently g belongs to $\Gamma^0_\beta(S)$.

For the converse, it suffices to show that $\Gamma^0_\beta(S) \subset \Gamma_\beta(S)$. Let g belong to $\Gamma^0_\beta(S)$. Choose a partition of unity ψ_{ij} subordinate to V^j_i. We can use a Poisson–type extension operator to extend $g\psi_{ij}$ to a function supported in V^j_i on one side of S. Each function, and hence their sum will satisfy (see Lemma I.4.5)

$$|X^\gamma 2^{j\beta} g \circ R_{-j}(y)| \leqslant C \max((1, d(y, A_0 \cap S_j))^{\beta-|\gamma|}$$

for all $y \in A_0 \cap \{w \mid r_j(w) > 0\}$. (Recall that X^γ is a monomial of length $|\gamma|$ in X_i and Y_i.) Let $x = R_{-j}(y)$ and divide both sides of the equation by $2^{j(\beta-|\gamma|)}$ to obtain

$$|X^\gamma g(x)| \leqslant C \max(1, d(x, S))^{\beta-|\gamma|} \text{ for } X \in A_j \cap \{w \mid r(w) > 0\}.$$

Therefore by Lemma I.4.6, g can be extended to be a function in Γ_β.

5. Boundary Regularity Near Characteristic Points

We begin by solving a dilation invariant problem. Denote $q(z) = \sum_{i,j=1}^{2n} a_{ij} x_i x_j$ ($z_j = x_j + i x_{j+n}$ and a_{ij} real numbers). Denote $D_q = \{(z, t \mid t > q(z)\}$. As in Section 3, we use polar coordinates (ρ, θ) on $\mathbb{H}^n$, with θ ranging over $S(1) = \{(z, t) \mid |z|^4 + t^2 = 1\}$ and $\rho = (|z|^4 + t^2)^{1/4}$. Let $\Omega_q = D_q \cap S(1)$. Here is the analogue of $g_0(x, t; v_0)\,\rho^{v_0}$ of Section 2.

(5.1) THEOREM. *If* 0 *is an isolated singular point of* D_q, *then there exists a critical index* $\lambda = \lambda_q > 0$ *and functions* $g_0(\theta)$, $v(\theta)$ *in* $C^\infty(\bar{\Omega}_q)$ *such that*

(a) $\mathscr{L}_0(g_0(\theta)\rho^\lambda) = 0$ on D_q.

(b) $g_0(\theta)\rho^\lambda = 0$ *on* ∂D_q, *i.e.*, $g_0(\theta) = 0$ *on* $\delta\Omega_q$.

(c) $g_0(\theta)\rho^\lambda > 0$ *on* D_q.

(d) $\mathscr{L}_0(g_0\rho^\lambda \log\rho + v\rho^\lambda) = 0$ *in* D_q.

(e) $\lambda_{q'}$ *increases to* λ_q *as* $\Omega_{q'}$ *shrinks to* Ω_q *or as* q' *increases to* q.

Proof. Let $g_0(\theta)$ be the first eigenfunction for the eigenvalue problem for Ω_q described in Section 3, and denoted $g_0(\theta)$, there. The properties (a), (b), and (c) listed here are proved there. The hypothesis that 0 is an isolated characteristic point for D_q is equivalent to the hypothesis that $\partial\Omega_q$ is noncharacteristic for the operator L_λ. It follows from [12, Theorem 3] that $g_0(\theta) \in C^\infty(\bar{\Omega}_q)$. For (d) note that

$$\mathscr{L}_0(g_0\rho^\lambda \log\rho + v\rho^\lambda)$$
$$= \mathscr{L}_0(g_0\rho^\lambda)\log\rho + (B_\lambda g_0)\rho^{\lambda-2} + (L_\lambda v)\rho^{\lambda-2} = 0$$

provided $L_\lambda v = -B_\lambda g_0$ for some second order operator B_λ. The solution v is far from unique. We can specify it by extending $-B_\lambda g_0$ to a smooth function h in a larger domain Ω^*, say, a rotation invariant one. Then solve the problem $L_\lambda v = g$ in Ω^*, $v = 0$ on $\partial\Omega^*$. This problem will have a solution provided $(n+\lambda)^2$ is not one of the associated eigenvalues for the L_λ Dirichlet problem in Ω^*. This is easy to arrange for appropriate Ω^*, since we know the eigenvalues' behavior from the zeros of Legendre functions. Again by Theorem 3 of [12], $v \in C^\infty(\bar{\Omega}^*) \subset C^\infty(\bar{\Omega}_q)$. The monotonicity in part (e) follows from the definition of the first eigenvalue as an infimum. The continuous dependence of λ_q on q follows from the fact that by Theorem 3 of [12], the first (normalized) eigenfunction has, say, $C^2(\bar{\Omega}_q)$ norm bounded by a constant depending only on the size of first few derivatives of q.

We associate to a strongly isolated characteristic point a critical index β_0 as follows. First translate the region D on the left so that the characteristic point coincides with the origin. As in Section 3, the defining function for D can be written $r(z,t) = t - q(z) + R(z,t)$, where $R(z,t) = O(|(z,t)|^3)$. The critical index is defined as the one given in Theorem 5.1 for D_q.

Let $\phi \in C_0^\infty(\mathbb{H}^n)$ be supported in a small neighborhood of a strongly isolated characteristic point of ∂D with critical index β_0.

(5.2) THEOREM. *If* $g \in \Gamma_\beta(\partial D)$, *then there is a unique* $u \in \Gamma_\varepsilon(\bar{D})$, *for some* $\varepsilon > 0$, *such that*

$$\mathscr{L}_0 u = 0, \qquad u|_{\partial G} = g.$$

If $\beta < \beta_0$, *then* $\phi u \in \Gamma_\beta(\bar{D})$. *If* $\beta \geqslant \beta_0$, ϕu *may fail to belong to* $\Gamma_{\beta_0}(\bar{D})$.

(5.3) THEOREM. *If* $g \in \Gamma_{\beta+2}(\partial D)$, $f \in \Gamma_\beta(\bar{D})$ *then there is a unique* $u \in \Gamma_\varepsilon(\bar{D})$, *for some* $\varepsilon > 0$, *such that* $\mathscr{L}_0 u = f$, $u|_{\partial D} = g$. *If* $\beta + 2 < \beta_0$, *then* $\phi u \in \Gamma_{\beta+2}(\bar{D})$.

The non-homogeneous version, Theorem (5.3), follows from Theorem (5.2). The additional fact one needs is that the global fundamental solution for $\mathscr{L}_0$ is smoothing of order 2 on Γ_β spaces. (See [4].)

(5.4) LEMMA. *Suppose that* S *is a* C^∞ *hypersurface in* $\mathbb{H}^n$ *with a strongly isolated singular point at* 0 *with critical index* β_0. *For any* $g \in C^\infty(S)$ *and any* $k < \beta_0$, *there exists a polynomial* P *on* $\mathbb{H}^n$ *of homogeneous degree* k *such that* $\mathscr{L}_0 P = 0$ *and* $|g - P| = O(|z|^{k+1})$ *on* S.

Proof. Let

$$\mathscr{P}_k(2n+1) = \{P \mid P \text{ is a polynomial homogeneous of degree } k\},$$

$$\mathscr{H}_k(2n+1) = \{P \mid P \in \mathscr{P}_k(2n+1) \text{ and } \mathscr{L}_0 P = 0\}.$$

Let $d_k(2n)$ denote the dimension of the space of polynomials of homogeneous in the ordinary sense of degree k on $\mathbb{R}^{2n}$.

A basis for $\mathscr{P}_k(2n+1)$ in monomials splits into those where t appears to the power, 0, 1, 2, etc. Thus

$$\dim \mathscr{P}_k(2n+1) = d_k(2n) + d_{k-2}(2n) + \cdots.$$

$\mathscr{L}_0$ maps $\mathscr{P}_k(2n+1)$ into $\mathscr{P}_{k-2}(2n+1)$. Hence, $\dim \mathscr{H}_k(2n+1) = \dim \ker \mathscr{L}_0 \geqslant \dim \mathscr{P}_k(2n+1) - \dim \mathscr{P}_{k-2}(2n+1) = d_k(2n)$. (One can actually show that $\mathscr{L}_0$ is surjective and hence $\dim \mathscr{H}_k(2n+1) = d_k(2n)$, but we will not need this.)

Let S_0 be the quadratic surface $t = q(z)$ tangent to S to third order at 0. Let $\mathscr{P}_k(S_0)$ denote homogeneous polynomials in x and y on S_0 of degree k. The restriction mapping $R : \mathscr{H}_k(2n+1) \to \mathscr{P}_k(S_0)$ is given by the substitution of $q(z)$ for t. Since $\dim \mathscr{H}_k(2n+1) \geqslant d_k(2n) = \dim \mathscr{P}_k(S_0)$, there are two possibilities:

(a) R is surjective

(b) There is a non-trivial element of $\mathscr{H}_k(2n+1)$ that vanishes identically on S_0.

Case (b) can only arise if $k \geqslant \beta_0$ (see Theorem 5.1). Thus for all $k < \beta_0$, R is surjective. In other words, the restrictions of $\mathscr{L}_0$-harmonic polynomials to S_0 span the polynomials on S_0 up to degree $k < \beta_0$. It follows that we can approximate any C^∞ function on S to order $|z|^{k+1}$ by members of $\mathscr{H}_j(2n+1)$ $j \leqslant k$, by an inductive procedure. This proves Lemma (5.4).

Let us pass to the proof of Theorem (5.2). Existence and uniqueness of a solution in $\Gamma_\epsilon(\bar{D})$ has been proved in Section 3. Therefore, we need only prove the estimate a priori

$$\|\phi u\|_{\Gamma_\beta(\bar{D})} \leqslant C \| g\|_{\Gamma_\beta(\partial D)}, \qquad \text{for} \quad g \in C^\infty(\partial D). \tag{5.5}$$

The fact that $\phi u \in \Gamma_\beta(\bar{D})$ for arbitrary g in $\Gamma_\beta(\partial D)$ follows by a routine limiting argument.

We will assume that the strongly isolated characteristic point is at the origin and denote $S - \partial D \cap \operatorname{supp} \phi$. We need only verify (5.5) in the case when β is not an integer. The corresponding estimate for integer values by real interpolation.

Suppose that $k < \beta < k+1$. By Lemma (5:4), we can choose a polynomial P such that $\mathscr{L}_0 P = 0$ and the Taylor series of g and P on S agree up to order k. It follows that $\| g - P\|_{\Gamma^0_\beta(S)} \leqslant C \| g\|_{\Gamma_\beta(S)}$. (See Section 4, and Part I, Appendix A.) Thus, replacing g by $g - P$, we may as well assume that $g \in \Gamma^0_\beta(S)$. Thus, with the notations of Section 4,

$$\| 2^{j\beta} g \circ R_{-j}\|_{\Gamma_\beta(A_0 \cap S_j)} \leqslant C, \tag{5.6}$$

uniformly as $j \to \infty$.

We will estimate u in two separate regions:

$$\text{(i)} \quad \bigcup_{i,j} G^j_i, \qquad \text{(ii)} \quad \bigcup_{i,j} V^j_i.$$

For any $\beta < \beta_0$, Theorem (5.1) implies that there exists a dilation invariant region D_q such that the quadratic form q is strictly smaller than the quadratic term of the defining function of D near 0 and such that β is the critical index for D_q, $D_q \supset D \cap \operatorname{supp} \phi$, and the associated barrier function on D_q satisfies

$$0 < c_1 \rho^\beta < g_\theta(\theta)\, \rho^\beta < c_2 \rho^\beta \qquad \text{for} \quad (\rho, \theta) \in S.$$

The (weak) maximum principle, applied in a ball centered at the origin intersected with D implies that $|u| \leqslant C g_0(\theta)\, \rho^\beta$ on $D \cap \operatorname{supp} \phi$, and hence $|u| \leqslant C\rho^\beta$ on $D \cap \operatorname{supp} \phi$.

This final estimate can be rewritten as

$$\| 2^{j\beta} u \circ R_{-j}\|_{L^\infty(R_j(V^j_i \cap D))} \leqslant C \tag{5.7}$$

with constants independent of i and j.

Finally, recall (3.9) and the estimate a priori from Part I (I.7.2) in a slightly weakened form says that of $\tilde{V}^j_i$, then

$$\begin{aligned}\| u \circ R_{-j}\|_{\Gamma_\beta(R_j(\tilde{V}^j_i \cap D))} \leqslant C(&\| u \circ R_{-j}\|_{L^\infty(R_j(V^j_i \cap D))} \\ &+ \| g \circ R_{-j}\|_{\Gamma_\beta(A_0 \cap S_j)}).\end{aligned}$$

When combined with (5.6) and (5.7) this yields

$$\| 2^{j\beta} u \circ R_{-j} \|_{\Gamma_\beta(R_j(V_i^j \cap D))} \leqslant C.$$

Similarly, the well-known interior estimate of Kohn that proves the hypoellipticity of $\mathscr{L}_0$ yields

$$\| 2^{j\beta} u \circ R_{-j} \|_{\Gamma_\beta(R_j(G_i^j))} \leqslant C.$$

Equation (5.5) now follows as in the proof of Theorem 4.3.

To see that the critical index β_0 is sharp in some sense, consider domains D such that $D \subset D_q$, D is tangent to D_q to third order. The situation is analogous to the one in Theorem (2.10); see Section 2. The function $g_0(\theta)\,\rho^{\beta_0}$ is a solution in D. $g_0(\theta)\,\rho^{\beta_0} \notin \Gamma_\beta(\bar{D})$ for $\beta > \beta_0$, but $g_0(\theta)\,\rho^{\beta_0} \in \Gamma_{\beta_0+1}(\partial D)$. (There are exceptions if β_0 is an even integer.) For failure at the critical index, use Theorem (5.1d). This also takes care of the case when β_0 is an even integer.

APPENDIX

Consider the Jacobi function $g_\nu(\tau)$ from Section 2. Let

$$y(\tau, \nu) = (1 - \tau^2)^{3/8} g_\nu(\tau) \qquad \text{(see [18, p. 67]}.$$

PROPOSITION 1. *Suppose that* $-1 < \tau < 1$ *and* $y(\tau, \nu) = 0$, *then* $(d/d\nu)\, y(\tau, \nu) \neq 0$, *i.e., the zero is simple, except possibly if* $\nu = -1/2$.

We will prove in Proposition 2 that the even and odd solutions to the equation (denoted g_0 and g_1 in Section 2) do not have a zero at $\nu = -1/2$. hence for our purposes that case does not arise.

To prove Proposition 1 note that y satisfies

$$\frac{d^2 y}{d\tau^2} + c(\nu)\, y = 0, \tag{1}$$

where $c(\nu) = \frac{1}{4}(\nu)(\nu + 1)/(1 - \tau^2)$.

We claim that

$$(c(\nu_1) - c(\nu)) \int_{-1}^{\tau} y(s, \nu_1)\, y(s, \nu)\, ds$$
$$= y(\tau, \nu) \frac{d}{d\tau}\, y(\tau, \nu) - y(\tau, \nu) \frac{d}{d\tau}\, y(\tau, \nu_1). \tag{2}$$

In fact, (1) implies that the derivative of the left and right hand sides of (2) with respect to τ are equal. It is easy to check, using the form of the solutions at $\tau = 1$ that both sides vanish at $\tau = 1$.

Now divide (2) by $(\nu_1 - \nu)$ and take the limit as $\nu_1 \to \nu$. Then

$$\int_1^{\tau_0} c'(\nu)\, y(s, \nu)^2\, ds = \left(\frac{d}{d\nu}\, y(\tau, \nu)\right)\left(\frac{d}{d\tau}\, y(\tau, \nu)\right).$$

Thus $(d/d\nu)\, y(\tau, \nu) \neq 0$ provided $c'(\nu) \neq 0$. Clearly $c'(\nu) = 0$ if and only if $\nu = -1/2$. As functions of z, g_0 and g_1 satisfy

$$Lg = \left(z(1-z)\frac{d^2}{dz^2} - \frac{3}{4}(2z-1)\frac{d}{dz} + q\right) g = 0,$$

where $q = \frac{1}{4}\nu(\nu + 1)$. Let $w = z^{1/4}$, then

$$L = \frac{1}{16}\, w^{-2} \left\{(1 - w^4)\frac{d^2}{dw^2} - 3w^3 \frac{d}{dw} + 16qw^2\right\}.$$

Let g be an even or odd solution to $Lg = 0$, $g = u + iv$.

PROPOSITION 2. *If there exists w_0, $0 < w_0 < 1$, such that $g(\pm w_0) = 0$, then ν is real and $\nu > 0$ or $\nu = -1$.*

Proof. Denote $d\mu(w) = (1 - w^4)^{-1/4} w^2\, dw$. Let $A = \operatorname{Re} L$. Then

$$\int_{w_0}^{w_0} (Af_1)\, f_2\, d\mu = \int_{-w_0}^{w_0} f_1 (Af_2)\, d\mu, \qquad \text{whenever } f_1(\pm w_0) = f_2(\pm w_0) = 0.$$

Let $c = 16 \operatorname{Im} q$. Taking real and imaginary parts of $Lg = 0$, we find $Au = cv$; $Av = -cu$. Therefore, $(Au)(\pm w_0) = 0$, and

$$-c^2 \int_{-w_0}^{w_0} u^2\, d\mu = \int_{-w_0}^{w_0} (A^2 u)\, u\, d\mu = \int_{-w_0}^{w_0} (Au)^2\, d\mu \geqslant 0.$$

Hence, $c = 0$. Thus

$$(2 \operatorname{Re} \nu + 1)(\operatorname{Im} \nu) = 4 \operatorname{Im} q = 0.$$

Therefore, either $\operatorname{Im} \nu = 0$, as desired, or $\operatorname{Re} \nu = -1/2$. Let us eliminate the case $\operatorname{Re} \nu = -1/2$.

If $\operatorname{Re} \nu = -1/2$, then q is real and $q = -\frac{1}{4} - (\operatorname{Im} \nu)^2 < 0$. So we may as well assume that g is real. Denote

$$\phi(w) = (1 - w^4)^{-1/4} w^2 \qquad \text{and} \qquad h(w) = (1 - w^4)^{3/4},$$

$$Lg = \tfrac{1}{16}\, \phi^{-1} (hg')' + qg.$$

Moreover, if $g(\pm w_0)=0$,

$$0=\int_{-w_0}^{w_0} (Lg)\, g\, d\mu = -\int_{-w_0}^{w_0} \tfrac{1}{16} h(g')^2\, dw + \int_{-w_0}^{w_0} qg^2\, d\mu.$$

This implies that

$$\int_{-w_0}^{w_0} qg^2\, d\mu \geqslant 0,$$

a contradiction because $q<0$.

Note that if $-1<\nu<0$, then it is also true that $q<0$, so there are no zeros in this case either. Finally, for $\nu=0$ or $\nu=-1$, $q=0$ and the equation can be solved explicitly and has no zeros. Let the zeros of the even solution be denoted $\nu_0(\tau)$, $\nu_2(\tau)$, $\nu_4(\tau)$, and the positive zeros of the odd solution by $\nu_1(\tau)$, $\nu_3(\tau)$, $\nu_5(\tau)$,.... At $\tau=0$ we computed that $\nu_{2k}(0)=4k+2$, $\nu_{2k+1}(0)=4k+3$, $k=0, 1, 2$,... (see Section 2). It is well-known that the functions $\nu_j(\tau)$ are continuous for $-1<\tau<1$. It can never happen that $\nu_{2k}(\tau)=\nu_{2k\pm 1}(\tau)=\nu$, because this would mean that two independent solutions to the equation $(d^2/d\tau^2)y+c(\nu)\,y=0$ vanish at an ordinary point τ. Hence interlacing inequalities

$$\nu_0(\tau)<\nu_1(\tau)<\nu_2(\tau)<\nu_3(\tau)<\cdots$$

are valid in the interval $-1<\tau<1$.

The symbol of the Poisson kernel in Section 2 was written in terms of the even and odd solutions to the Jacobi equation above. Recall that $\tau=\cos\theta$; denote

$$z=\tfrac{1}{2}(1-\cos\theta), \qquad z_0=\tfrac{1}{2}(1-\cos\theta_0).$$

The odd and even parts of the symbol are

$$\sigma_0(\theta,\nu)=F_0(z,\nu)/F_0(z_0,\nu),$$
$$\sigma_1(\theta,\nu)=(\operatorname{sgn}\theta)\,|z|^{1/4}F_1(z,\nu)/|z_0|^{1/4}F_1(z_0,\nu);$$

where F_0 and F_1 are hypergeometric series

$$F_0(z,\nu)=F((\nu+1)/2, -\nu/2\,; 3/4\,; z).$$
$$F_1(z,\nu)=F(\nu/2+3/4, -\nu/2+1/4; 5/4\,; z).$$

The zeros analyzed in Propositions 1 and 2 are the poles of σ_0 and σ_1.

PROPOSITION 3. *Let $\nu=\beta+i\eta$. Assume that ν is a bounded distance from the poles of $\sigma_j(\theta,\nu)$. Then for $j=0, 1$,*

(a) $|\sigma_j(\theta, \nu)| \leqq Ce^{-|\theta-\theta_0|\,|\eta|}$,

(b) $|\theta-\theta_0|^\delta \left|\dfrac{\partial^l}{\partial\nu^l}\dfrac{\partial^k}{\delta\theta^k}\sigma_0(\theta,\nu)\right| \leqq C(1+|\eta|)^{-\delta-l+k}$,

(c) $|\theta-\theta_0|^\delta \left|\dfrac{\partial^l}{\partial\nu^l}\dfrac{\partial^k}{\partial\theta^k}\left(\dfrac{\sigma_1(\theta,\nu)}{(\operatorname{sgn}\theta)\,|z|^{1/4}}\right)\right| \leqslant C(1+|\eta|)^{-\delta-l+k}$.

Note that (b) differs from (c) only because we have removed the factor $(\operatorname{sgn}\theta)\,|z|^{1/4}$ from σ_1 because it is not smooth as a function of θ as $\theta\to 0$. It is smooth as a function of x and t. (See Sect. 2.) $\theta=0$ corresponds to the positive t axis.

Proposition 3 is an easy consequence of an asymptotic expansion for hypergeometric functions due to Watson [19].

$$\begin{aligned}|F(\alpha+i\eta,\beta-i\eta;\gamma;\tfrac{1}{2}(1-\cos\theta))&\\ -|\eta|^{1/2-\gamma}(c_1(\theta)\,e^{\eta\theta}+c_2(\theta)\,e^{-\eta\theta})|&\\ \leqslant C\,|\eta|^{-\gamma}\,e^{|\eta\theta|}\,\theta^{-l},&\end{aligned} \tag{3}$$

where l, c_1, c_2, and C depend on α, β, and γ. $c_1(\theta)$ and $c_2(\theta)$ are continuous and non-zero for $\theta\neq 0$ and $c_j(\theta)=O(\theta^{-l})$ as $\theta\to 0$.

An examination of the proof shows that the asymptotic expansion can be differentiated. In particular,

$$\begin{aligned}\left|\frac{\partial^j}{\partial\eta^j}\frac{\partial^k}{\partial\theta^k}F(\alpha+i\eta,\beta-i\eta;\gamma;\tfrac{1}{2}(1-\cos\theta))\right|&\\ \leqq C\,|\eta|^{1/2-\gamma-j+k}\theta^{-l}e^{|\eta\theta|},&\end{aligned} \tag{4}$$

where C and l depend on j, k, α, β, γ. Estimate (3) implies Proposition 3a for θ near θ_0, namely,

$$|\sigma_j(\theta,\beta+i\eta)| \leqq Ce^{-|\theta-\theta_0|\,|\eta|}.$$

Therefore, $|\theta-\theta_0|^\delta\,|\sigma_j| \leqq C(1+|\eta|)^{-\delta}$, uniformly as $\theta\to\theta_0$. To take care of values of θ near zero we need a trivial estimate that follows from the power series expansion for F:

$$\left|\frac{\partial^l}{\partial\eta^l}\frac{\partial^k}{\partial\theta^k}F(\alpha+i\eta,\beta-i\eta;\gamma;\tfrac{1}{2}(1-\cos\theta))\right| \leqq C \tag{5}$$

whenever $|\theta|<\varepsilon_0(1+|\eta|)^{-1}$. This estimate completes the proof of (a) in the trivial range θ near 0. In a similar way, (4) and (5) yield (b) and (c).

Correction to Sections 3 and 5

Equation 3.3 is false. There is an additional non-Hermitian term to the quadratic form $(L_\lambda \phi, \phi)$. As a result several corrections must be made, which we list here. We will use the notations of the paper.

The validity of Theorem 5.1 is in doubt because the critical index β_0 is defined by a non-self-adjoint eigenvalue problem (see [20]). In particular, we do not know if β_0 exists or if it is real. Therefore, we cannot deduce Theorems 5.2 and 5.3 using the maximum principle. A complete analysis that avoids the maximum principle and that is also valid for the Dirichlet problem for $\mathscr{L}_\alpha$, $\alpha \neq 0$, and $\Box_b$ on CR manifolds will appear in [20].

A weakened version of the main theorems 5.2 and 5.3 still follows from the same method because Theorem 5.1 is valid for rotation invariant regions. In this case the barrier functions are the Jacobi functions $g_\lambda(\tau)$ introduced in Section 3. The correct statement is

THEOREM 5.1′. *Let $M \in \mathbb{R}$, $D_M = \{(z, t): t > M|z|^2\}$, and $\bar{\Omega}_M = \{(z, t): t \geqslant M|z|^2;\ |z|^4 + t^2 = 1\}$. There is a critical index $\lambda = \lambda_M > 0$ and functions g, v in $C^\infty(\bar{\Omega}_M)$ satisfying*

(i) $\mathscr{L}_0(g(\theta)\rho^\lambda) = 0$ *in* D_M; $g(\theta)\rho^\lambda = 0$ *on* ∂D_M.

(ii) $g(\theta)\rho^\lambda > 0$ *in* D_M.

(iii) $\mathscr{L}_0(g(\theta)\rho^\lambda \log \rho + v(\theta)\rho^\lambda) = 0$ *in* D_M.

(iv) λ_M *increases as* M *increases.*

The corrected version of 5.2 is

THEOREM 5.2′. *Let $0 \in \partial D$ be a strongly isolated characteristic point and $r(z, t) = t - q(z) + R(z, t)$ be a defining function for ∂D with $R(z, t) = O(|(z, t)|^3)$. Let $\phi \in C_0^\infty(\mathbb{H}^n)$ be supported in a small neighborhood of 0. Choose M so that $q(z) \geqslant M|z|^2$. If $g \in \Gamma_\beta(\partial D)$, then there is a unique $u \in \Gamma_\beta(\bar{D})$ for some $\varepsilon > 0$ such that $\mathscr{L}_0 u = 0$ in D; $u|_{\partial D} = g$. If $\beta < \lambda_M$, then $\phi u \in \Gamma_\beta(\bar{D})$. If $\beta \geqslant \lambda_M$ and $q(z) = M|z|^2$, then ϕu may fail to belong to $\Gamma_\beta(\bar{D})$. There is a similar correction to 5.3.*

None of the other theorems need to be changed, but Lemma 3.2 must be replaced by

LEMMA 3.2′. *For any spherical cap $\omega \subset S(1)$, there exist β, c_1, c_2, $c_3 > 0$ and a barrier function $g(\rho, \theta)$ continuous for $\theta \in S(1)\backslash\omega$, $1 \geqslant \rho \geqslant 0$ and satisfying*

(a) $\mathscr{L}_0 g = 0$ *for* $\theta \in S(1)\backslash\bar{\omega}$, $1 > \rho > 0$.

(b) $c_1 < g(\rho, \theta)/g(\rho, \theta') < c_2$, *for* $1 \geqslant \rho > 0$, $\theta, \theta' \in S(1)\backslash\omega$.

(c) $g(\rho_2, \theta)/g(\rho_1, \theta') \leqslant c_3(\rho_2/\rho_1)^\beta$, $\theta, \theta' \in S(1)\backslash\omega$, $1 \geqslant \rho_1 \geqslant \rho_2 \geqslant 0$.

Proof. Let ω' be a non-empty open set, $\omega' \subset\subset \omega$. Let $\Omega = S(1)\backslash\bar{\omega}$. Consider the following function $g(\rho, \theta)$ given by the Perron process: the largest g satisfying $\mathscr{L}_0 g = 0$ for $\theta \in \Omega$, $1 > \rho > 1$ and $g(1, \theta) \leqslant 1$, $\theta \in \bar{\Omega}$; $g(\rho, \theta) \leqslant 0$, $0 \leqslant \rho < 1$, $\theta \in \partial\Omega$. g is non-negative, and continuous at all boundary points except the non-smooth ones, that is, $\rho = 1$, $\theta = 1$, $\theta \in \partial\Omega$ and possibly at $\rho = 0$. Part (b) follows from Bony's Harnack Principle [1] and dilation invariance of $\mathscr{L}_0$.

We will now show that $g(\rho, \theta) \leqslant c_1 \rho^\beta$. Denote $\sigma = (\rho, \theta)$. Consider the measure $d\mu_\sigma$ on $S(1)$ representing the linear functional $f \mapsto u(\sigma)$, where f is a continuous function on $S(1)$ and u solves $\mathscr{L}_0 u = 0$ in the unit ball $\rho < 1$ and has boundary values f on $S(1)$. Thus $u(\sigma) = \int_{S(1)} f d\mu_\sigma$. Gaveau [5] has calculated explicitly the formula for $d\mu_\sigma$ when $\sigma = 0$. In fact, it is a smooth measure that vanishes at just two points. In particular, $d\mu_0(\omega') > 0$. By Bony's Harnack Principle, $\varepsilon = \inf\{d\mu_\sigma(\omega') : |\sigma| \leqslant \frac{1}{2}\}$ is positive. The maximum principle in the region $\{(\rho, \theta) = \rho < 1, \Theta \in \Omega\}$ implies that $g(\sigma) \leqslant (1 - d\mu_\sigma(\omega'))$. Thus $|g(\sigma)| \leqslant 1 - \varepsilon$ for $|\sigma| \leqslant \frac{1}{2}$. The same argument applied to the balls $\rho < 2^{-k}$, $h = 1, 2, \ldots$, using a dilation of $d\mu_\sigma(\omega')$, proves $g(\sigma) \leqslant (1 - \varepsilon)^h$ for $|\sigma| \leqslant 2^{-k}$. Part (c) then follows from the same argument using the balls $\rho < \rho_1 2^{-k}$, $k = 0, 1, 2, \ldots$. (Theorem 3.1 follows from Lemma 3.2′ by a standard barrier argument in the nature of [10].)

Finally, the remark preceding 3.9 may be replaced by

REMARK. *Let V be a neighborhood in $\mathbb{H}^n$ of a non-characteristic point of ∂D with local coordinates. (x, y): $V \to \mathbb{R}^{2n} \times \mathbb{R}$ so that $y > 0$ corresponds to $V \cap D$. If $f \in C^2(D)$, and u satisfies $\mathscr{L}_0 u = 0$ in D, $u|_{\partial D} = f$, then $|u(x, y) - f(x_0) - f'(x_0)(x - x_0)| \leqslant c(y + |x - x_0|)$ for $y > 0$.*

Proof. A suitable (Heisenberg group) translation and dilation of $b(z, t) = |(z_0, t_0)|^{-2n} - |(z_0, t_0)^{-1}(z, t)|^{-2n}$ so that (z_0, t_0) coincides with $(x_0, 0)$ satisfies (in local coordinates)

(i) $\mathscr{L}_0\, b = 0$ near $(x_0, 0)$.

(ii) $c_1(|x - x_0|^2 + y^2) \leqslant b(x, y) \leqslant c_2(|x - x_0| + y)$ for $y > 0$.

There is a function $F(x, y)$ that is a first degree polynomial in (z, t) such that $|f(x_0) + f'(x_0)(x - x_0) - F(x, y)| \leqslant C(|x - x_0|^2 + y^2)$. Since $\mathscr{L}_0 F = 0$, the maximum principle implies that $|u - F| \leqslant Cb$, and the remark follows. (Proposition 3.9 follows from the new remark as easily as from the old.)

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